On approximation of quadratic irrationals by rational numbers

Let \( \xi = [0, \overline{a_1, \ldots, a_n}] \) be a quadratic irrational with discriminant \( D \) and let \( p_i/q_i \) be its \( i \)th convergent. Put

\[
K_i = \left| \xi - \frac{p_i}{q_i} \right|^2.
\]

We first suppose that \( n = 1 \), i.e. \( \xi = [0, \overline{a_1}] \).

**Statement 1.** We have:

\[
\lim_{i \to \infty} K_i = \frac{1}{\sqrt{D}}.
\]

Moreover, \( K_1 < K_3 < K_5 < \ldots \) and \( K_2 > K_4 > K_6 > \ldots \).

We now generalize Statement 1 to the case \( n > 1 \). Let \( r \) be an integer with \( 0 \leq r \leq n - 1 \). We define \( n \)-tuples \((a_1^{(r)}, \ldots, a_{n-1}^{(r)})\) in the following way:

\[
\begin{align*}
(a_0^{(0)}, \ldots, a_{n-1}^{(0)}) &= (a_2, a_3, \ldots, a_n) \\
(a_1^{(1)}, \ldots, a_{n-1}^{(1)}) &= (a_1, a_2, \ldots, a_n) \\
(a_1^{(2)}, a_{n-1}^{(2)}) &= (a_n, a_1, \ldots, a_{n-2}) \\
(a_1^{(3)}, \ldots, a_{n-1}^{(3)}) &= (a_{n-1}, a_n, \ldots, a_{n-4}, a_{n-3}) \\
& \vdots \\
(a_1^{(n-1)}, \ldots, a_{n-1}^{(n-1)}) &= (a_3, a_4, \ldots, a_{n-1}).
\end{align*}
\]

Denote by \( N(r) \) the numerator of the continued fraction \([a_1^{(r)}, \ldots, a_{n-1}^{(r)}]\).

**Statement 2.** If \( n \) is even, then

\[
N(0)a_1 + \sum_{r=1}^{n-1} (-1)^r N(r) a_{n-r+1} = 0.
\]

**Statement 3.** If \( n \) is even, \( i > n \) and \( i \equiv 0 \) or \( n - 1 \mod n \), then

\[
N(1)q_{2i-n} = q_iq_{i-1} - q_iq_{i-n}q_{i-n-1}.
\]

Put

\[
G = \gcd(N(0), \ldots, N(n-1)), \quad F(k, r) = \frac{C(k)N(0)^k N(r)^{k+1}}{G^{2k+1}D^k \sqrt{D}},
\]

where \( C(k) = \frac{(2k)!}{k!(k+1)!} \) are the Catalan numbers.

**Statement 4.** For any \( r \) with \( 0 \leq r \leq n - 1 \) we have

\[
\lim_{j \to \infty} K_{nj+r} = F(0, r).
\]

Moreover, a subsequence of \( \{K_{nj+r}\} \) with odd indexes (if any) is increasing and a subsequence of \( \{K_{nj+r}\} \) with even indexes (if any) is decreasing.

**Corollary 5.** If \( n \) is even, then

\[
\lim_{j \to \infty} \left( K_{nj}a_1 + \sum_{r=1}^{n-1} (-1)^r K_{nj+r} a_{n-r+1} \right) = 0.
\]

**Statement 6.** Let \( r \) be an integer as in Statement 4. Let also \( i \) be an odd integer \( \geq 1 \) and let \( m \) be an integer \( \geq 0 \). Suppose \( i \equiv r \mod n \). Then

\[
\sum_{k=0}^{2m+1} (-1)^k \frac{F(k, r)}{q_i^{2k+2}} < p_i q_i - \xi < \sum_{k=0}^{2m} (-1)^k \frac{F(k, r)}{q_i^{2k+2}}.
\]

**Corollary 7.** Let \( i, m \) and \( r \) be integers as in Statement 6. Then

\[
\left| \xi - \frac{p_i}{q_i} - \sum_{k=0}^{m} (-1)^{k+1} \frac{F(k, r)}{q_i^{2k+2}} \right| < \frac{F(m+1, r)}{q_i^{2m+4}}.
\]