M340L - Matrices and Matrix Calculations - Fall 2002

Mid-Term TEST #2, November 7, 2002

I. (30 points) Let $A = \begin{bmatrix} 1 & -3 & 1 & -7 & -5 \\ -1 & 4 & 0 & 10 & 6 \\ 2 & -3 & 5 & -5 & -7 \end{bmatrix}$.

(a) Find a spanning set for Col A;
(b) Find a basis for Col A;
(c) Find the dimension of Col A;
(d) Find a spanning set for Nul A;
(e) Find a basis for Nul A;
(f) Find the dimension of Nul A;
(g) Find the rank of A.
PROVIDE CLEAR EXPLANATIONS.

II. (30 points)

1. Evaluate the determinant $D = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 6 & 9 & 1 & 1 \\ -1 & -2 & -3 & -4 & 3 \\ 2 & 4 & 7 & 1 & -2 \\ 1 & 3 & 7 & -8 & 1 \end{vmatrix}$.

2. Use Cramer's rule to solve the system $\begin{cases} 7x + 2y = 3 \\ 11x - 5y = -1. \end{cases}$

3. Find the area of the triangle whose vertices are $(-3, 2),\ (0, 0),\ (1, 5)$.

4. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$.

III. (20 points) Let $V = \mathbb{P}_3$. Determine if the given set is a subspace of V. Justify your answer.

(a) All polynomials of the form $\mathbf{p}(t) = at^3 - bt + 2c$, where a, b, c are in $\mathbb{R}$.

(b) All polynomials of the form $\mathbf{p}(t) = at^3 - 2t^2 + c$, where a, b, c are in $\mathbb{R}$.

(c) All polynomials of the form $\mathbf{p}(t) = at^4 + b$, where a, b are in $\mathbb{R}$.

(d) All polynomials of the form $\mathbf{p}(t) = at^3 + bt^2 + ct + d$, where a, b, c, d are negative numbers.

IV. (20 points) Let $V = \mathbb{P}_2$. Let also

$$\mathbf{p_1}(t) = -t - t^2, \quad \mathbf{p_2}(t) = 2t^2 - 5t, \quad \mathbf{p_3}(t) = 2t + t^2, \quad \mathbf{p_4}(t) = t^2 + 3t - 1, \quad \mathbf{q}(t) = -2t + t^2 - 5.$$

(a) Determine whether the vectors $\mathbf{p_1}(t),\ \mathbf{p_2}(t),\ \mathbf{p_3}(t)$ form a basis for V. If "yes", find coordinates of $\mathbf{q}(t)$ in this basis. Explain your work and justify your conclusions.

(b) Determine whether the vectors $\mathbf{p_1}(t),\ \mathbf{p_2}(t),\ \mathbf{p_4}(t)$ form a basis for V. If "yes", find coordinates of $\mathbf{q}(t)$ in this basis. Explain your work and justify your conclusions.

(c) Determine whether the vectors $\mathbf{p_2}(t),\ \mathbf{p_3}(t),\ \mathbf{p_4}(t)$ span V. Explain your work and justify your conclusions.

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SOLUTIONS

I. (a) The spanning set is:

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -7 \\ 10 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix}.$$

(b) To find the basis for Col A we first row reduce the matrix A to obtain the reduced echelon form:

$$A = \begin{bmatrix} 1 & -3 & 1 & -7 & -5 \\ -1 & 4 & 0 & 10 & 6 \\ 2 & -3 & 5 & -5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & -7 & -5 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 3 & 3 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & -7 & -5 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & -7 & -5 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4 & 2 & -2 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

According to the pivot columns, the basis for Col A is:

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}.$$

(c) Since the basis of Col A contains two vectors, its dimension is two.

(d) To find the spanning set we should solve the system $A\mathbf{x} = \mathbf{0}$. By (b) we have:

$$\begin{cases} x_1 + 4x_3 + 2x_4 - 2x_5 = 0 \\ \quad x_2 + x_3 + 3x_4 + x_5 = 0, \end{cases}$$

therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_3 - 2x_4 + 2x_5 \\ -x_3 - 3x_4 - x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the spanning set is:

$$\begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(e)=(d)

(f) Since the basis of Nul A contains three vectors, its dimension is three.

(g) The rang of A is equal to the dimension of Col A and therefore is two.

II. 1.

$$D = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 6 & 9 & 1 & 1 \\ -1 & -2 & -3 & -4 & 3 \\ 2 & 4 & 7 & 1 & -2 \\ 1 & 3 & 7 & -8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & -11 & -14 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & -7 & -12 \\ 0 & 1 & 4 & -16 & -9 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & -16 & -9 \\ 0 & 0 & 1 & -7 & -12 \\ 0 & 0 & 0 & -11 & -14 \\ 0 & 0 & 0 & 0 & 8 \end{vmatrix} = 88.$$

2. We have

$$x = \frac{\begin{vmatrix} 3 & 2 \\ -1 & -5 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 11 & -5 \end{vmatrix}} = \frac{13}{57}, \qquad y = \frac{\begin{vmatrix} 7 & 3 \\ 11 & -1 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 11 & -5 \end{vmatrix}} = \frac{40}{57}.$$

3. The area A is equal to the absolute value of

$$\frac{1}{2} \cdot \begin{vmatrix} -3 & 1 \\ 2 & 5 \end{vmatrix} = -\frac{17}{2}.$$

So, $A = 17/2$.

4.

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

III. (a) Yes. In fact, first of all, this is a subset of $\mathbb{P}_3$. Furthermore,

(i) $\mathbf{0} = 0 \cdot t^3 - 0 \cdot t + 2 \cdot 0;$

(ii) $\mathbf{p_1}(t) + \mathbf{p_2}(t) = a_1 t^3 - b_1 t + 2c_1 + a_2 t^3 - b_2 t + 2c_2$

$$= (a_1 + a_2)t^3 - (b_1 + b_2)t^2 + 2(c_1 + c_2) = \mathbf{p_3}(t);$$

(iii) $c \cdot \mathbf{p_1}(t) = c \cdot a_1 t^3 - c \cdot b_1 t + 2c \cdot c_1 = \mathbf{p_4}(t).$

(b) No. There is no zero in this set.

(c) No. This is not a subset.

(d) No. There is no zero in this set.

IV.

Step 1. Consider the standard basis of $\mathbb{P}_2$, that is $B = \{1,\ t,\ t^2\}$.

Step 2. Find the coordinates of $\mathbf{p_1}(t),\ \mathbf{p_2}(t),\ \mathbf{p_3}(t),\ \mathbf{p_4}(t)$, and $\mathbf{q}(t)$ in this basis. We have:

$$[\mathbf{p_1}]_B = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \quad [\mathbf{p_2}]_B = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \quad [\mathbf{p_3}]_B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad [\mathbf{p_4}]_B = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad [\mathbf{q}]_B = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}.$$

(a) The vectors $\mathbf{p_1}(t),\ \mathbf{p_2}(t),\ \mathbf{p_3}(t)$ DO NOT form a basis for V, since a number of pivot positions of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -5 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

is less than the dimension of $\mathbb{P}_2$. In fact,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -5 & 2 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 2 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 2 \\ 0 & 7 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, there are 2 pivot positions, whereas the dimension of $\mathbb{P}_2$ is three.

(b) The vectors $\mathbf{p_1}(t),\ \mathbf{p_2}(t),\ \mathbf{p_4}(t)$ FORM a basis for V, since a number of pivot positions of the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -5 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

is equal to the dimension of $\mathbb{P}_2$, namely 3. In fact,

$$A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -5 & 3 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 3 \\ 0 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 3 \\ 0 & 0 & -1 \\ 0 & 7 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 3 \\ 0 & 7 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

To find the coordinates of $\mathbf{q}(t)$ in this basis, we row reduce an augmented matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & -5 \\ -1 & -5 & 3 & -2 \\ -1 & 2 & 1 & 1 \end{bmatrix}.$$

We have

$$A \sim \begin{bmatrix} -1 & -5 & 3 & -2 \\ 0 & 0 & -1 & -5 \\ -1 & 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 3 & -2 \\ 0 & 0 & -1 & -5 \\ 0 & 7 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 3 & -2 \\ 0 & 7 & -2 & 3 \\ 0 & 0 & -1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -5 & 0 & -17 \\ 0 & 7 & 0 & 13 \\ 0 & 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} -1 & -5 & 0 & -17 \\ 0 & 1 & 0 & 13/7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & -54/17 \\ 0 & 1 & 0 & 13/7 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

So, $\mathbf{q}(t)$ has coordinates $(54/17,\ 13/7,\ 5)$.

(c) The vectors $\mathbf{p_2}(t)$, $\mathbf{p_3}(t)$, $\mathbf{p_4}(t)$ SPAN V, since a number of pivot positions of the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ -5 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

is equal to the dimension of $\mathbb{P}_2$, namely 3. In fact,

$$A \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ -5 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & 5 \\ 0 & 0 & -1 \\ -10 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & 5 \\ 0 & 0 & -1 \\ 0 & 9 & 11 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & 5 \\ 0 & 9 & 11 \\ 0 & 0 & -1 \end{bmatrix}.$$