M340L - Matrices and Matrix Calculations - Fall 2002

Mid-Term TEST #1, October 3, 2002

I. (20 points)

1. Find all solutions to the system of linear equations:

$$\begin{cases} 2x_1 + 3x_2 + 11x_3 + 5x_4 = 2 \\ x_1 + x_2 + 5x_3 + 2x_4 = 1 \\ x_1 + x_2 + 3x_3 + 4x_4 = -3 \\ 6x_1 + 7x_2 + 29x_3 + 15x_4 = 2 \end{cases}$$

2. Indicate the following items:
   (a) Augmented matrix;
   (b) Echelon form;
   (c) Reduced echelon form;
   (d) Pivot positions;
   (e) Basic variables;
   (f) Free variables.
3. Write the solution set in the parametric form and indicate the following items:
   (a) Particular solution of the system;
   (b) General solution of a correspondent homogeneous system.

II. (10 points)

1. Let

$$A = \begin{bmatrix} 1 & 8 \\ 0 & 9 \\ -7 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -7 \\ 0 & 1 \\ 6 & -1 \end{bmatrix}.$$

Find $A + B,\ -2B,\ A - 3B$.

2. Let

$$A = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -2 & 3 \end{bmatrix}.$$

Find AB and BA.

3. Let

$$A = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ 3 & -1 & 5 \end{array}\right], \quad B = \left[\begin{array}{cc} 0 & 7 \\ \hline 3 & 4 \\ 1 & 0 \end{array}\right].$$

The 3 columns of A are partitioned into a set of 2 columns. The 3 rows of B are partitioned into a set of 2 rows. Find AB using block multiplication.

III. (20 points)

1. Rewrite the following system in the matrix form:

$$\begin{cases} 2x_1 - 4x_2 + x_3 = 3 \\ x_1 - 5x_2 + 3x_3 = -1 \\ x_1 - x_2 + x_3 = 1 \end{cases}$$

2. Solve this system using the inverse matrix.

IV. (20 points)

Suppose an economy has three sectors, Chemical, Metal, and Power. Chemical sells 40% of its output to Metal and 20% to Power and retains the rest. Metal sells 20% of its output to Chemical and 30% to Power and retains the rest. Power sells 40% of its output to Chemical and 50% to Metal and retains the rest.

(a) Construct the exchange table for the economy.

(b) Find a set of equilibrium prices for the economy.

(c) Find a set of equilibrium prices when the price for the Power is 100 units.

V. (30 points)
Let

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} -7 \\ 3 \\ -7 \end{bmatrix}, \quad \mathbf{v_4} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Is $\mathbf{b}$ in $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$?

(b) Are vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ linearly dependent or independent?

(c) Do the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ span $\mathbb{R}^3$?

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SOLUTIONS

I. First, we row reduce the augmented matrix:

$$\begin{bmatrix} 2 & 3 & 11 & 5 & 2 \\ 1 & 1 & 5 & 2 & 1 \\ 1 & 1 & 3 & 4 & -3 \\ 6 & 7 & 29 & 15 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & 2 & 1 \\ 1 & 1 & 3 & 4 & -3 \\ 2 & 3 & 11 & 5 & 2 \\ 6 & 7 & 29 & 15 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & 2 & 1 \\ 0 & 0 & -2 & 2 & -4 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 3 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 3 & -4 \\ 0 & 0 & -2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 5 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 & -4 \\ 0 & 0 & -2 & 2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 & -7 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix, echelon form, reduced echelon form and pivot positions are indicated.
The basic variables are $x_1,\ x_2$, and x_3. The free variable is x_4.
We now write the solution set of the system in the parametric form:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7-5x_4 \\ -2-2x_4 \\ 2+x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -2 \\ 1 \\ 1 \end{bmatrix},$$

where

$$\mathbf{p} = \begin{bmatrix} -7 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

is the particular solution of the system and

$$\mathbf{v}_h = x_4 \begin{bmatrix} -5 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

is the general solution of the correspondent homogeneous system.

II. 1. We have:

$$A+B=\begin{bmatrix} 1 & 8 \\ 0 & 9 \\ -7 & 1 \end{bmatrix}+\begin{bmatrix} -1 & -7 \\ 0 & 1 \\ 6 & -1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 10 \\ -1 & 0 \end{bmatrix},$$

$$-2B=-2\begin{bmatrix} -1 & -7 \\ 0 & 1 \\ 6 & -1 \end{bmatrix}=\begin{bmatrix} 2 & 14 \\ 0 & -2 \\ -12 & 2 \end{bmatrix},$$

$$A-3B=\begin{bmatrix} 1 & 8 \\ 0 & 9 \\ -7 & 1 \end{bmatrix}-3\begin{bmatrix} -1 & -7 \\ 0 & 1 \\ 6 & -1 \end{bmatrix}=\begin{bmatrix} 4 & 29 \\ 0 & 6 \\ -25 & 4 \end{bmatrix}.$$

2. We have:

$$AB=\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}\begin{bmatrix} 5 & -2 & 3 \end{bmatrix}=\begin{bmatrix} 3\cdot 5 & 3\cdot(-2) & 3\cdot 3 \\ 4\cdot 5 & 4\cdot(-2) & 4\cdot 3 \\ 2\cdot 5 & 2\cdot(-2) & 2\cdot 3 \end{bmatrix}=\begin{bmatrix} 15 & -6 & 9 \\ 20 & -8 & 12 \\ 10 & -4 & 6 \end{bmatrix},$$

$$BA=\begin{bmatrix} 5 & -2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}=[5\cdot 3+(-2)\cdot 4+3\cdot 2]=[13].$$

3. We have:

$$AB=\left[\begin{array}{c|cc} 1 & 1 & 0 \\ 3 & -1 & 5 \end{array}\right]\left[\begin{array}{cc} 0 & 7 \\ \hline 3 & 4 \\ 1 & 0 \end{array}\right]=\begin{bmatrix} A_1 & A_2 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=A_1B_1+A_2B_2,$$

where

$$A_1=\begin{bmatrix} 1 \\ 3 \end{bmatrix},\quad A_2=\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix},\quad B_1=\begin{bmatrix} 0 & 7 \end{bmatrix},\quad B_2=\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} AB=A_1B_1+A_2B_2= & \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 0 & 7 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \\ = & \begin{bmatrix} 1\cdot 0 & 1\cdot 7 \\ 3\cdot 0 & 3\cdot 7 \end{bmatrix}+\begin{bmatrix} 1\cdot 3+0\cdot 1 & 1\cdot 4+0\cdot 0 \\ (-1)\cdot 3+5\cdot 1 & (-1)\cdot 4+5\cdot 0 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 7 \\ 0 & 21 \end{bmatrix}+\begin{bmatrix} 3 & 4 \\ 2 & -4 \end{bmatrix} \\ = & \begin{bmatrix} 3 & 11 \\ 2 & 17 \end{bmatrix}. \end{aligned}$$

III. 1. We first rewrite the system in the matrix form:

$$\begin{bmatrix} 2 & -4 & 1 \\ 1 & -5 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

2. To solve this system we first consider an augmented matrix $[A, I]$ and transform it as $[I, A^{-1}]$ using row operations:

$$\begin{bmatrix} 2 & -4 & 1 & 1 & 0 & 0 \\ 1 & -5 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 & 0 & 1 & 0 \\ 2 & -4 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 & 0 & 1 & 0 \\ 0 & 6 & -5 & 1 & -2 & 0 \\ 0 & 4 & -2 & 0 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & 0 & 1 & 0 \\ 0 & 6 & -5 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 & -1/2 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & -1/2 & 1/2 \\ 0 & 6 & -5 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & -1/2 & 1/2 \\ 0 & 0 & -2 & 1 & -1/2 & -3/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 0 & -1/4 & 1/4 \\ 0 & 0 & 1 & -1/2 & 1/4 & 3/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1/2 & 0 & -1/4 & 5/4 \\ 0 & 1 & -1/2 & 0 & -1/4 & 1/4 \\ 0 & 0 & 1 & -1/2 & 1/4 & 3/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1/4 & -3/8 & 7/8 \\ 0 & 1 & 0 & -1/4 & -1/8 & 5/8 \\ 0 & 0 & 1 & -1/2 & 1/4 & 3/4 \end{bmatrix},$$

therefore

$$A^{-1} = \begin{bmatrix} 1/4 & -3/8 & 7/8 \\ -1/4 & -1/8 & 5/8 \\ -1/2 & 1/4 & 3/4 \end{bmatrix}.$$

Since

$$\mathbf{x} = A^{-1}\mathbf{b},$$

we get

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4 & -3/8 & 7/8 \\ -1/4 & -1/8 & 5/8 \\ -1/2 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 + 3/8 + 7/8 \\ -3/4 + 1/8 + 5/8 \\ -3/2 - 1/4 + 3/4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

IV. (a) The exchange table is:

C	M	P	
.4	.2	.4	C
.4	.5	.5	M
.2	.3	.1	P

(b) To find the set of equilibrium prices we should solve the following system of equations:

$$\begin{cases} .6x_1 - .2x_2 - .4x_3 = 0 \\ -.4x_1 + .5x_2 - .5x_3 = 0 \\ -.2x_1 - .3x_2 + .9x_3 = 0. \end{cases}$$

We have:

$$\begin{bmatrix} .6 & -.2 & -.4 & 0 \\ -.4 & .5 & -.5 & 0 \\ -.2 & -.3 & .9 & 0 \end{bmatrix} \sim \begin{bmatrix} -.2 & -.3 & .9 & 0 \\ .6 & -.2 & -.4 & 0 \\ -.4 & .5 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} -.2 & -.3 & .9 & 0 \\ 0 & -1.1 & 2.3 & 0 \\ 0 & 1.1 & -2.3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -.2 & -.3 & .9 & 0 \\ 0 & -1.1 & 2.3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 & -9/2 & 0 \\ 0 & 1 & -23/11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -15/11 & 0 \\ 0 & 1 & -23/11 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{bmatrix} x_C \\ x_M \\ x_P \end{bmatrix} = x_P \begin{bmatrix} 15/11 \\ 23/11 \\ 1 \end{bmatrix}.$$

(c) If we put $x_P = 100$, we get

$$\begin{bmatrix} x_C \\ x_M \\ x_P \end{bmatrix} = \begin{bmatrix} 1500/11 \\ 2300/11 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 136.36 \\ 209.09 \\ 100 \end{bmatrix}.$$

V. (a) The answer is "Yes". In fact, by a definition, the vector $\mathbf{b}$ is in Span$(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$ if and only if the equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} + x_4\mathbf{v_4} = \mathbf{b}$$

has a solution. The correspondent linear system is consistent if and only if an echelon form of its augmented matrix has no row of the form

$$[\,0\ \ldots\ 0\ c\,] \text{ with } c \text{ nonzero.}$$

We have

$$\begin{bmatrix} 1 & 3 & -7 & -2 & 1 \\ 0 & -1 & 3 & 1 & 1 \\ -2 & 1 & -7 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -7 & -2 & 1 \\ 0 & -1 & 3 & 1 & 1 \\ 0 & 7 & -21 & -8 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -7 & -2 & 1 \\ 0 & -1 & 3 & 1 & 1 \\ 0 & 0 & 0 & -1 & 10 \end{bmatrix}.$$

The last matrix has no row of the form

$$[\,0\ \ldots\ 0\ c\,] \text{ with } c \text{ nonzero,}$$

therefore the vector $\mathbf{b}$ is in Span$(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$.

(b) The vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent. In fact, $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent if and only if the equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} = \mathbf{0}$$

has a nontrivial solution. This equation has a nontrivial solution if and only if the correspondent homogeneous equation $A\mathbf{x} = 0$ has at least one free variable. The equation $A\mathbf{x} = 0$ has at least one free variable if and only if a number of pivot positions of the matrix A is less than a number of the columns. In other words, the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent if and only if the number of pivot positions of the matrix $[\ \mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ ]$ is less than the number of vectors. We have:

$$\begin{bmatrix} 1 & 3 & -7 \\ 0 & -1 & 3 \\ -2 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -7 \\ 0 & -1 & 3 \\ 0 & 7 & -21 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -7 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the number of pivot positions is 2 and the number of vectors is 3, $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent

(c) The answer is "Yes". In fact, the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ span $\mathbb{R}^3$ if and only if the matrix $[\ \mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}\ ]$ has a pivot position in every row, which is true by (a).