PROBLEM:

Let $\bar{u}$ and $\bar{y}$ be nonzero vectors in R^n. Find vectors $\hat{y}$ and $\bar{z}$ such that

$$\bar{y} = \hat{y} + \bar{z},$$

where $\hat{y}$ is a multiple of $\bar{u}$ and $\bar{z}$ is orthogonal to $\bar{u}$.

DEFINITION:

The vector

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}$$

is called the orthogonal projection of $\bar{y}$ onto $\bar{u}$ and denoted by

$$\text{proj}_{\bar{u}} \bar{y}.$$

The vector $\bar{z}$ is called the component of $\bar{y}$ orthogonal to $\bar{u}$.

EXAMPLE:

Let $\bar{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ **and** $\bar{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$**. Find the orthogonal projection of** $\bar{y}$ **onto** $\bar{u}$**.**

SOLUTION:

We first find the orthogonal projection of $\bar{y}$ **onto** $\bar{u}$**. We have**

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u} = \frac{7 \cdot 4 + 6 \cdot 2}{4 \cdot 4 + 2 \cdot 2} \bar{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

REMARK:

Note that the orthogonal projection of $\bar{y}$ onto $\bar{u}$ is exactly the same as the orthogonal projection of $\bar{y}$ onto $c\bar{u}$, where c is any nonzero scalar. Hence this projection is determined by the subspace L spanned by $\bar{u}$. Therefore sometimes we denote $\hat{y}$ by

$$\mathrm{proj}_L \bar{y}.$$

So,

$$\hat{y} = \mathrm{proj}_{\bar{u}} \bar{y} = \bar{y} \mathrm{proj}_L \bar{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}.$$

THEOREM(The Orthogonal Decomposition Theorem):

Let W be a subspace of R^n. Then each $\bar{y}$ in R^n can be written uniquely in the form

$$\bar{y} = \hat{y} + \bar{z},$$

where $\hat{y}$ is in W and $\bar{z}$ is in $W^{\perp}$. In fact, if $\{\bar{u}_1, \ldots, \bar{u}_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \ldots + \frac{\bar{y} \cdot \bar{u}_p}{\bar{u}_p \cdot \bar{u}_p}\bar{u}_p$$

and $\bar{z} = \bar{y} - \hat{y}$.

DEFINITION:

The vector $\hat{y}$ is called the orthogonal projection of $\bar{y}$ onto W and written as

$$\text{proj}_W \bar{y}.$$

EXAMPLE:

Let

$$\bar{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \bar{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(i) Find the orthogonal projection of $\bar{y}$ onto $W = \text{Span}\{\bar{u}_1, \bar{u}_2\}$;

(ii) Write $\bar{y}$ as the sum of a vector in W and a vector orthogonal to W.

SOLUTION:

(i) By the Theorem above, the orthogonal projection of $\bar{y}$ onto W is

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2}\bar{u}_2$$

$$= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

(ii) By the Theorem above we have $\bar{z} = \bar{y} - \hat{y}$, therefore

$$\bar{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}.$$

So,

$$\bar{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}.$$

PROBLEM:

Let

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \bar{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}.$$

(i) Find the orthogonal projection of $\bar{y}$ onto $W = \text{Span}\{\bar{u}_1, \bar{u}_2\}$;

(ii) Write $\bar{y}$ as the sum of a vector in W and a vector orthogonal to W.

SOLUTION:

(i) By the Theorem above, the orthogonal projection of $\bar{y}$ onto W is

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2}\bar{u}_2$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

(ii) By the Theorem above we have $\bar{z} = \bar{y} - \hat{y}$, therefore

$$\bar{z} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

So,

$$\bar{y} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

PROBLEM:

Let

$$\boldsymbol{A} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \bar{\boldsymbol{b}} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Find a solution of $A\bar{x} = \bar{b}$.

<u>DEFINITION</u>:

Let A be an $m \times n$ matrix and $\bar{b}$ be in R^m. The <u>general least-squares problem</u> is the problem of finding an $\bar{x}$ that makes

$$\|\bar{b} - A\bar{x}\|$$

as small as possible. A <u>least-squares solution</u> of $A\bar{x} = \bar{b}$ is an $\hat{x}$ in R^n such that

$$\|\bar{b} - A\hat{x}\| \leq \|\bar{b} - A\bar{x}\|$$

for all $\bar{x}$ in R^n.

THEOREM:

The set of least-squares solutions of $A\bar{x} = \bar{b}$ coincides with the nonempty set of solutions of the system

$$A^T A\bar{x} = A^T\bar{b}.$$

We usually call this system the normal equations.

EXAMPLE:

Let

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Find a least-squares solution of the inconsistent system $A\bar{x} = \bar{b}$.

SOLUTION:

We first compute A^TA and $A^T\bar{b}$. We have

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and

$$A^T\bar{b} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Then the equation $A^TA\bar{x} = A^T\bar{b}$ becomes

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix},$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

PROBLEM:

Let

$$\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \quad \bar{\boldsymbol{b}} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Find a least-squares solution of the inconsistent system $\boldsymbol{A\bar{x} = \bar{b}}$**.**

SOLUTION:

We first compute A^TA and $A^T\bar{b}$. We have

A^TA

$$= \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

and

$$A^T\bar{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}.$$

Then the equation $A^TA\bar{x} = A^T\bar{b}$ becomes

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix},$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$