THEOREM:

Let $\mathcal{S} = \{\bar{u}_1, \ldots, \bar{u}_p\}$ be an orthogonal basis for a subspace W of R^n. For each $\bar{y}$ in W the weights in the linear combination

$$\bar{y} = c_1\bar{u}_1 + \ldots + c_p\bar{u}_p$$

are given by

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \ldots, p).$$

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in $\mathcal{S}$.

SOLUTION:

We have:

$$\bar{y} \cdot \bar{u}_1 = 11, \ \bar{y} \cdot \bar{u}_2 = -12, \ \bar{y} \cdot \bar{u}_3 = -33$$

and

$$\bar{u}_1 \cdot \bar{u}_1 = 11, \ \bar{u}_2 \cdot \bar{u}_2 = 6, \ \bar{u}_3 \cdot \bar{u}_3 = 33/2,$$

so

$$c_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\bar{y} \cdot \bar{u}_3}{\bar{u}_3 \cdot \bar{u}_3} = \frac{-33}{33/2} = -2$$

therefore

$$[\bar{x}]_{\mathcal{S}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

THEOREM (The Gram-Schmidt Process):

Given an arbitrary basis $\{\bar{x}_1, \ldots, \bar{x}_p\}$ for a subspace W of R^n, define

$$\bar{v}_1 = \bar{x}_1$$

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\bar{v}_2$$

$$\ldots$$

$$\bar{v}_p = \bar{x}_p - \frac{\bar{x}_p \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \ldots - \frac{\bar{x}_p \cdot \bar{v}_{p-1}}{\bar{v}_{p-1} \cdot \bar{v}_{p-1}}\bar{v}_{p-1}$$

Then $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is an orthogonal basis for W.

EXAMPLE:

Let

$$\bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

is the basis for a subspace W of R^4. Find an orthogonal basis for W.

SOLUTION:

Step 1: Put

$$\bar{v}_1 = \bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Step 2: Put

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

Step 3: Put

$$\bar{\boldsymbol{v}}_3 = \bar{\boldsymbol{x}}_3 - \frac{\bar{\boldsymbol{x}}_3 \cdot \bar{\boldsymbol{v}}_1}{\bar{\boldsymbol{v}}_1 \cdot \bar{\boldsymbol{v}}_1}\bar{\boldsymbol{v}}_1 - \frac{\bar{\boldsymbol{x}}_3 \cdot \bar{\boldsymbol{v}}_2}{\bar{\boldsymbol{v}}_2 \cdot \bar{\boldsymbol{v}}_2}\bar{\boldsymbol{v}}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

THEOREM:

Let $\mathcal{S} = \{\bar{u}_1, \ldots, \bar{u}_p\}$ be an orthogonal basis for a subspace W of R^n. For each $\bar{y}$ in W the weights in the linear combination

$$\bar{y} = c_1\bar{u}_1 + \ldots + c_p\bar{u}_p$$

are given by

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \ldots, p).$$

PROOF:

Let $c_1, \dots, c_p$ be such numbers that

$$\bar{y} = c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_p\bar{u}_p. \qquad (*)$$

If we multiply both sides of $(*)$ by $\bar{u}_1$, we get

$\bar{y} \cdot \bar{u}_1$

$= c_1\bar{u}_1 \cdot \bar{u}_1 + c_2\bar{u}_2 \cdot \bar{u}_1 + \dots + c_p\bar{u}_p \cdot \bar{u}_1$

$= c_1\bar{u}_1 \cdot \bar{u}_1 + 0 + \dots + 0$

$= c_1\bar{u}_1 \cdot \bar{u}_1$

because of orthogonality of $\bar{u}_1, \dots, \bar{u}_p$. So, $\bar{y} \cdot \bar{u}_1 = c_1\bar{u}_1 \cdot \bar{u}_1$, therefore

$$c_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}.$$

Similarly, if we multiply both sides of $(*)$ by $\bar{u}_j$, we deduce

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \dots, p).$$

PROBLEM:

Let $\bar{u}$ and $\bar{y}$ be nonzero vectors in R^n. Find vectors $\hat{y}$ and $\bar{z}$ such that

$$\bar{y} = \hat{y} + \bar{z},$$

where $\hat{y}$ is a multiple of $\bar{u}$ and $\bar{z}$ is orthogonal to $\bar{u}$.

SOLUTION:

Rewrite $\bar{y} = \hat{y} + \bar{z}$, as $\bar{z} = \bar{y} - \hat{y}$ and multiply both sides by $\bar{u}$:

$$\bar{z} \cdot \bar{u} = (\bar{y} - \hat{y}) \cdot \bar{u}$$

But $\bar{z}$ is orthogonal to $\bar{u}$, therefore

$$0 = (\bar{y} - \hat{y}) \cdot \bar{u}. \qquad (*)$$

Since $\hat{y}$ is a multiple of $\bar{u}$, we have

$$\hat{y} = \alpha \bar{u}, \text{ where } \alpha \text{ is a scalar.}$$

Substituting this into $(*)$, we get

$$0 = (\bar{y} - \alpha \bar{u}) \cdot \bar{u} = \bar{y} \cdot \bar{u} - \alpha \bar{u} \cdot \bar{u},$$

hence

$$\alpha = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \quad \text{and} \quad \hat{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}.$$

DEFINITION:

The vector $\hat{y}$ is called the orthogonal projection of $\bar{y}$ onto $\bar{u}$ and denoted by

$$\text{proj}_{\bar{u}} \bar{y}.$$

The vector $\bar{z}$ is called the component of $\bar{y}$ orthogonal to $\bar{u}$.

EXAMPLE:

Let $\bar{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $\bar{y}$ onto $\bar{u}$. Write $\bar{y}$ as a sum of two orthogonal vectors, one in Span $\{\bar{u}\}$ and one orthogonal to $\bar{u}$.

SOLUTION:

We first find the orthogonal projection of $\bar{y}$ onto $\bar{u}$. We have

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u} = \frac{7 \cdot 4 + 6 \cdot 2}{4 \cdot 4 + 2 \cdot 2} \bar{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

We now find the component $\bar{z}$. We have

$$\bar{z} = \bar{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Finally, we write $\bar{y}$ as a sum of two orthogonal vectors, one in Span $\{\bar{u}\}$ and one orthogonal to $\bar{u}$:

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

<u>REMARK</u>:

Note that the orthogonal projection of $\bar{y}$ onto $\bar{u}$ is exactly the same as the orthogonal projection of $\bar{y}$ onto $c\bar{u}$, where c is any nonzero scalar. Hence this projection is determined by the subspace L spanned by $\bar{u}$. Therefore sometimes we denote $\hat{y}$ by

$$\mathrm{proj}_L \bar{y}.$$

So,

$$\hat{y} = \mathrm{proj}_{\bar{u}} \bar{y} = \bar{y} \mathrm{proj}_L \bar{y} = \frac{\bar{y} \cdot \bar{u}}{\bar{u} \cdot \bar{u}} \bar{u}.$$

<u>THEOREM</u>(The Orthogonal Decomposition Theorem):

Let W be a subspace of R^n. Then each $\bar{y}$ in R^n can be written uniquely in the form

$$\bar{y} = \hat{y} + \bar{z},$$

where $\hat{y}$ is in W and $\bar{z}$ is in $W^{\perp}$. In fact, if $\{\bar{u}_1, \ldots, \bar{u}_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \ldots + \frac{\bar{y} \cdot \bar{u}_p}{\bar{u}_p \cdot \bar{u}_p}\bar{u}_p$$

and $\bar{z} = \bar{y} - \hat{y}$.