DEFINITION:

If $\bar{u}$ and $\bar{v}$ are vectors in R^n, then $\bar{u}^T\bar{v}$ is called the inner product (or dot product) of $\bar{u}$ and $\bar{v}$ and written as

$$\bar{u} \cdot \bar{v}$$

REMARK:

In other words, if

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

then

$$\bar{u} \cdot \bar{v} = \bar{u}^T\bar{v} = [u_1 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1v_1 + \ldots + u_nv_n.$$

DEFINITION:

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal (perpendicular) if

$$\bar{u} \cdot \bar{v} = 0.$$

EXAMPLE:

Vectors $\bar{u} = (5, -3, 1)$ and $\bar{v} = (6, 9, -3)$ are orthogonal, since

$$\bar{u} \cdot \bar{v} = 5 \cdot 6 + (-3) \cdot 9 + 1 \cdot (-3) = 0.$$

DEFINITION:

A set of vectors $\{\bar{u}_1, \ldots, \bar{u}_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is

$$\bar{u}_i \cdot \bar{u}_j = 0$$

for any $i \neq j$.

EXAMPLE:

Let

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Then $\{\bar{u}_1, \ \bar{u}_2, \ \bar{u}_3\}$ is an orthogonal set.

PROBLEM:

Let

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Show that $\{\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3\}$ is an orthogonal set.

SOLUTION:

We have

$$\bar{u}_1 \cdot \bar{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$$

$$\bar{u}_1 \cdot \bar{u}_3 = 3\left(-\frac{1}{2}\right) + 1(-2) + 1\left(\frac{7}{2}\right) = 0$$

$$\bar{u}_2 \cdot \bar{u}_3 = -1\left(-\frac{1}{2}\right) + 2(-2) + 1\left(\frac{7}{2}\right) = 0$$

THEOREM:

If $\mathcal{S} = \{\bar{u}_1, \ldots, \bar{u}_p\}$ is an orthogonal set of nonzero vectors in R^n, then $\mathcal{S}$ is linearly independent and hence is a basis (so-called, an orthogonal basis) for the subspace spanned by $\mathcal{S}$. Of course, if $p = n$, then $\mathcal{S}$ is a basis for R^n.

EXAMPLE:

Let $\mathcal{S} = \{\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Then $\mathcal{S}$ is an orthogonal basis for R^3.

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in $\mathcal{S}$.

SOLUTION:

We have:

$$\begin{bmatrix} 3 & -1 & -1/2 & 6 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & 7/2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & -1 & -1/2 & 6 \\ 1 & 1 & 7/2 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -7 & 5.5 & 3 \\ 0 & -1 & 5.5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 5.5 & -9 \\ 0 & -7 & 5.5 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 5.5 & -9 \\ 0 & 0 & -33 & 66 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -5.5 & 9 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

THEOREM:

Let $\mathcal{S} = \{\bar{u}_1, \ldots, \bar{u}_p\}$ be an orthogonal basis for a subspace W of R^n. For each $\bar{y}$ in W the weights in the linear combination

$$\bar{y} = c_1\bar{u}_1 + \ldots + c_p\bar{u}_p$$

are given by

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \ldots, p).$$

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix},\ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in $\mathcal{S}$.

SOLUTION:

We have:

$\bar{y} \cdot \bar{u}_1 = 11, \ \bar{y} \cdot \bar{u}_2 = -12, \ \bar{y} \cdot \bar{u}_3 = -33$

and

$\bar{u}_1 \cdot \bar{u}_1 = 11, \ \bar{u}_2 \cdot \bar{u}_2 = 6, \ \bar{u}_3 \cdot \bar{u}_3 = 33/2,$

so

$$c_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\bar{y} \cdot \bar{u}_3}{\bar{u}_3 \cdot \bar{u}_3} = \frac{-33}{33/2} = -2$$

therefore

$$[\bar{x}]_{\mathcal{S}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

DEFINITION:

A set of vectors $\{\bar{u}_1, \ldots, \bar{u}_p\}$ in R^n is said to be an orthonormal set if it is an orthogonal set of unit vectors.

A set of vectors $\{\bar{u}_1, \ldots, \bar{u}_p\}$ in R^n is said to be an orthonormal basis if it is an orthogonal basis of unit vectors.

EXAMPLE:

Let

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\{\bar{e}_1, \ \bar{e}_2, \ \bar{e}_3\}$ is the orthonormal basis for R^3.

EXAMPLE:

We know that

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \ \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

is the orthogonal basis for R^3. Then

$$\bar{w}_1 = \frac{\bar{u}_1}{\|\bar{u}_1\|} = \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\bar{w}_2 = \frac{\bar{u}_2}{\|\bar{u}_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\bar{w}_3 = \frac{\bar{u}_3}{\|\bar{u}_3\|} = \sqrt{\frac{2}{33}} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{1}{\sqrt{66}} \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$$

is the orthonormal basis for R^3.

THEOREM (The Gram-Schmidt Process):

Given an arbitrary basis $\{\bar{x}_1, \ldots, \bar{x}_p\}$ for a subspace W of R^n, define

$$\bar{v}_1 = \bar{x}_1$$

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\bar{v}_2$$

$$\ldots$$

$$\bar{v}_p = \bar{x}_p - \frac{\bar{x}_p \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \ldots - \frac{\bar{x}_p \cdot \bar{v}_{p-1}}{\bar{v}_{p-1} \cdot \bar{v}_{p-1}}\bar{v}_{p-1}$$

Then $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is an orthogonal basis for W.

EXAMPLE:

Let

$$\bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

is the basis for a subspace W of R^4. Find an orthogonal basis for W.

SOLUTION:

Step 1: Put

$$\bar{v}_1 = \bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Step 2: Put

$$\bar{v}_2 = \bar{x}_2 - \frac{\bar{x}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

Step 3: Put

$$\bar{v}_3 = \bar{x}_3 - \frac{\bar{x}_3 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\bar{v}_1 - \frac{\bar{x}_3 \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\bar{v}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$