

DEFINITION:

If \bar{u} and \bar{v} are vectors in R^n , then $\bar{u}^T \bar{v}$ is called the inner product (or dot product) of \bar{u} and \bar{v} and written as

$$\bar{u} \cdot \bar{v}$$

REMARK:

In other words, if

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

then

$$\begin{aligned} \bar{u} \cdot \bar{v} &= \bar{u}^T \bar{v} = [u_1 \ \dots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= u_1 v_1 + \dots + u_n v_n. \end{aligned}$$

DEFINITION:

Two vectors \bar{u} and \bar{v} in R^n are orthogonal (perpendicular) if

$$\bar{u} \cdot \bar{v} = 0.$$

EXAMPLE:

Vectors $\bar{u} = (5, -3, 1)$ and $\bar{v} = (6, 9, -3)$ are orthogonal, since

$$\bar{u} \cdot \bar{v} = 5 \cdot 6 + (-3) \cdot 9 + 1 \cdot (-3) = 0.$$

DEFINITION:

A set of vectors $\{\bar{u}_1, \dots, \bar{u}_p\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is

$$\bar{u}_i \cdot \bar{u}_j = 0$$

for any $i \neq j$.

EXAMPLE:

Let

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Then $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is an orthogonal set.

PROBLEM:

Let

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Show that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is an orthogonal set.

SOLUTION:

We have

$$\bar{u}_1 \cdot \bar{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$$

$$\bar{u}_1 \cdot \bar{u}_3 = 3 \left(-\frac{1}{2} \right) + 1(-2) + 1 \left(\frac{7}{2} \right) = 0$$

$$\bar{u}_2 \cdot \bar{u}_3 = -1 \left(-\frac{1}{2} \right) + 2(-2) + 1 \left(\frac{7}{2} \right) = 0$$

THEOREM:

If $\mathcal{S} = \{\bar{u}_1, \dots, \bar{u}_p\}$ is an orthogonal set of nonzero vectors in R^n , then \mathcal{S} is linearly independent and hence is a basis (so-called, an orthogonal basis) for the subspace spanned by \mathcal{S} . Of course, if $p = n$, then \mathcal{S} is a basis for R^n .

EXAMPLE:

Let $\mathcal{S} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Then \mathcal{S} is an orthogonal basis for R^3 .

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in \mathcal{S} .

SOLUTION:

We have:

$$\begin{aligned} & \begin{bmatrix} 3 & -1 & -1/2 & 6 \\ 1 & 2 & -2 & 1 \\ 1 & 1 & 7/2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & -1 & -1/2 & 6 \\ 1 & 1 & 7/2 & -8 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -7 & 5.5 & 3 \\ 0 & -1 & 5.5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 5.5 & -9 \\ 0 & -7 & 5.5 & 3 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & 5.5 & -9 \\ 0 & 0 & -33 & 66 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -5.5 & 9 \\ 0 & 0 & 1 & -2 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \end{aligned}$$

THEOREM:

Let $\mathcal{S} = \{\bar{u}_1, \dots, \bar{u}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . For each \bar{y} in W the weights in the linear combination

$$\bar{y} = c_1\bar{u}_1 + \dots + c_p\bar{u}_p$$

are given by

$$c_j = \frac{\bar{y} \cdot \bar{u}_j}{\bar{u}_j \cdot \bar{u}_j} \quad (j = 1, \dots, p).$$

PROBLEM:

Let $\mathcal{S} = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$, where

$$\bar{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \bar{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}.$$

Find coordinates of $\bar{y} = (6, 1, -8)$ in \mathcal{S} .

SOLUTION:

We have:

$$\bar{y} \cdot \bar{u}_1 = 11, \quad \bar{y} \cdot \bar{u}_2 = -12, \quad \bar{y} \cdot \bar{u}_3 = -33$$

and

$$\bar{u}_1 \cdot \bar{u}_1 = 11, \quad \bar{u}_2 \cdot \bar{u}_2 = 6, \quad \bar{u}_3 \cdot \bar{u}_3 = 33/2,$$

so

$$c_1 = \frac{\bar{y} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1} = \frac{11}{11} = 1$$

$$c_2 = \frac{\bar{y} \cdot \bar{u}_2}{\bar{u}_2 \cdot \bar{u}_2} = \frac{-12}{6} = -2$$

$$c_3 = \frac{\bar{y} \cdot \bar{u}_3}{\bar{u}_3 \cdot \bar{u}_3} = \frac{-33}{33/2} = -2$$

therefore

$$[\bar{x}]_{\mathcal{S}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

DEFINITION:

A set of vectors $\{\bar{u}_1, \dots, \bar{u}_p\}$ in R^n is said to be an orthonormal set if it is an orthogonal set of unit vectors.

A set of vectors $\{\bar{u}_1, \dots, \bar{u}_p\}$ in R^n is said to be an orthonormal basis if it is an orthogonal basis of unit vectors.

EXAMPLE:

Let

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is the orthonormal basis for R^3 .