DEFINITION:

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following 10 axioms (or rules):

1. The sum of $\bar{u}$ and $\bar{v}$, denoted by $\bar{u}+\bar{v}$, is in V.

2. $\bar{u}+\bar{v}=\bar{v}+\bar{u}$.

3. $(\bar{u}+\bar{v})+\bar{w}=\bar{u}+(\bar{v}+\bar{w})$.

4. There is a zero vector $\bar{0}$ in V such that $\bar{u}+\bar{0}=\bar{u}$.

5. For each $\bar{u}$ in V, there is a vector $-\bar{u}$ in V such that $\bar{u}+(-\bar{u})=\bar{0}$.

6. The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.

7. $c(\bar{u}+\bar{v})=c\bar{u}+c\bar{v}$.

8. $(c+d)\bar{u}=c\bar{u}+d\bar{u}$.

9. $c(d\bar{u})=(cd)\bar{u}$.

10. $1\cdot\bar{u}=\bar{u}$.

These axioms must hold for all vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ in V and all scalars c and d.

EXAMPLE:

1. $$R^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in R \right\}$$

2. The set P_n of polynomials of degree at most n:

$$\bar{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable t are real numbers.

3. The set of all real-valued functions defined on R.

DEFINITION:

A subspace of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.

2. H is closed under vector addition. That is, for each $\bar{u}$ and $\bar{v}$ in H, the sum $\bar{u} + \bar{v}$ is in H.

3. H is closed under multiplication by scalars. That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H.

EXAMPLE:

The set consisting of only the zero vector $\bar{0}$ in a vector space V is a subspace of V, called the zero subspace and written as $\{\bar{0}\}$.

WARNING:

R^2 is not a subspace of R^3, because R^2 is not a subset of R^3. However, P_2 is a subspace of P_3.

THEOREM:

If $\bar{v}_1, \ldots, \bar{v}_p$ are in a vector space V, then Span $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a subspace of V.

EXAMPLE:

Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

By the Theorem above

$$\text{Span}\{\bar{v}_1, \bar{v}_2\}$$

is a subspace of R^3.

EXAMPLE:

The set

$$H=\left\{\begin{bmatrix}4a-b\\2b\\a-2b\\a-b\end{bmatrix}:a,b\in R\right\}$$

is a subspace of R^4, because

$$\begin{bmatrix}4a-b\\2b\\a-2b\\a-b\end{bmatrix}=a\underbrace{\begin{bmatrix}4\\0\\1\\1\end{bmatrix}}_{\bar{v}_1}+b\underbrace{\begin{bmatrix}-1\\2\\-2\\-1\end{bmatrix}}_{\bar{v}_2}.$$

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is not a vector space.

SOLUTION:

H is not a vector space, since $\bar{0} \notin H$ (the second entry is always nonzero).

DEFINITION:

Let A be an $m \times n$ matrix.

1. The null space of A, written as Nul A, is the set of all solutions to the homogeneous equation $A\bar{x} = \bar{0}$.

2. The row space of A, written as Row A, is the set of all linear combinations of the row vectors of A.

3. The column space of A, written as Col A, is the set of all linear combinations of the columns of A. So, if $A = [\bar{a}_1 \ldots \bar{a}_n]$, then Col $A = \text{Span}\{\bar{a}_1, \ldots, \bar{a}_n\}$.

REMARK:

Nul A and Row A are subspaces of R^n, whereas Col A is a subspace of R^m.

EXAMPLE:

Find a spanning set for the column space, row space, and null space of the matrix

$$A=\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

SOLUTION:

(a) Obviously, columns of A, i.e.

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \quad \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix}$$

form the spanning set for Col A.

(b) Obviously, rows of A, i.e.

$$(-3, \quad 6, \ -1, \ 1, \ -7)$$
$$(1, \ -2, \quad 2, \ 3, \ -1)$$
$$(2, \ -4, \quad 5, \ 8, \ -4)$$

form the spanning set for the row space of A.

(c) To find a spanning set for Nul $\boldsymbol{A}$, we find the general solution of $\boldsymbol{A\bar{x} = \bar{0}}$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0, \end{cases}$$

$$\text{so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}},$$

so Nul $\boldsymbol{A}$ =Span $\{\bar{u}, \bar{v}, \bar{w}\}$.

DEFINITION:

Let H be a subspace of a vector space V. A set of vectors

$$\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_p\}$$

in V is a basis for H if

(a) $\mathcal{B}$ is a linearly independent set;

(b) $H = \text{Span}\,\{\bar{b}_1, \ldots, \bar{b}_p\}$.

STANDARD BASIS FOR R^n:

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

STANDARD BASIS FOR P_n:

Vectors

$$\bar{e}_1 = 1,\ \bar{e}_2 = t,\ \bar{e}_3 = t^2, \ldots,\ \bar{e}_{n+1} = t^n$$

form the so-called standard basis for the vector space P_n.

TEST 1:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are linearly independent if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has p pivots.

TEST 2:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ span R^n if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has n pivots.

TEST 3:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ form a basis of R^n if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has n pivots and $p = n$.

EXAMPLE:

The set of vectors

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

form a basis for R^3, since

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

and we have 3 vectors and 3 pivots.

EXAMPLE:

The set of vectors

$$3+7t,\ 5+t-2t^3,\ t-2t^2,\ 1+16t-6t^2+2t^3$$

do not form a basis for P_3, since

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & 0 & 1 \\ 0 & 32 & -3 & -41 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we have 3 pivots and 4 columns.

SOLUTION (DETAILS):

Let $\mathcal{B} = \{1,\ t,\ t^2,\ t^3\}$ be the standard basis of P_3. Then polynomials

$3+7t,\ 5+t-2t^3,\ t-2t^2,\ 1+16t-6t^2+2t^3$

produce coordinate vectors

$$\begin{bmatrix} 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 16 \\ -6 \\ 2 \end{bmatrix}$$

relative to $\mathcal{B}$. We have:

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & 0 & 1 \\ 0 & 32 & -3 & -41 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are 3 pivots and 4 columns, the polynomials

$3+7t,\ 5+t-2t^3,\ t-2t^2,\ 1+16t-6t^2+2t^3$

do not form a basis for P_3.

<u>EXAMPLE</u>:

Find bases for the row space, the column space, and the null space of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Since the first two rows have pivots, they form a basis for the row space of A.

(b) Since the first two columns have pivots, they form a basis for Col A.

(c) Finally, for Nul A we need the reduced echelon form. We have:

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding system is

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

so $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is the basis for Nul A.

DEFINITION:

Suppose $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis for a vector space V and $\bar{x}$ is in V. The coordinates of $\bar{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

EXAMPLE:

Let

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Find coordinates of $\bar{x}$ relative to the basis $\mathcal{B} = \{\bar{b}_1, \ \bar{b}_2\}$.

SOLUTION:

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix},$$

therefore

$$c_1 = -2 \quad \text{and} \quad c_2 = 3,$$

so

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

EXAMPLE:

Let $\mathcal{E} = \{1,\ t,\ t^2\}$ be the standard basis for P_2. Find coordinates of the vector

$$\bar{p}(t) = -4 + 3t - 5t^2$$

relative to $\mathcal{E}$.

SOLUTION:

By the definition above we have:

$$[\bar{p}]_{\mathcal{E}} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}.$$

EXAMPLE:

Determine whether the polynomials

$$1+t,\ 1+t^2,\ t+t^2$$

form a basis for P_2. If "Yes", find coordinates of the vector

$$\bar{p}(t) = -4 + 3t - 5t^2$$

relative to this basis.

SOLUTION:

Let $\mathcal{E} = \{1,\ t,\ t^2\}$ be the standard basis of P_2. Then polynomials

$$1+t,\ 1+t^2,\ t+t^2$$

produce coordinate vectors

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\1\end{bmatrix}$$

relative to $\mathcal{E}$. We have:

$$\begin{bmatrix}1&1&0\\1&0&0\\0&1&1\end{bmatrix} \sim \begin{bmatrix}1&1&0\\0&-1&0\\0&0&1\end{bmatrix}.$$

Since there are 3 pivots and 3 columns, the polynomials

$$1+t,\ 1+t^2,\ t+t^2$$

form a basis for P_2.

Let

$$\mathcal{B} = \{1 + t,\ 1 + t^2,\ t + t^2\}.$$

To find coordinates of the vector

$$\bar{p}(t) = -4 + 3t - 5t^2$$

relative to $\mathcal{B}$, we consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

therefore

$$[\bar{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}.$$

THEOREM:

Let $\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ and $\mathcal{C} = \{\bar{c}_1, \ldots, \bar{c}_n\}$ be bases of a vector space V. Then there is a unique matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\bar{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\bar{x}]_{\mathcal{B}},$$

where

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [[\bar{b}_1]_{\mathcal{C}} \ \ [\bar{b}_2]_{\mathcal{C}} \ \ \ldots \ \ [\bar{b}_n]_{\mathcal{C}}].$$

REMARK:

One can show that

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$$

EXAMPLE:

Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$ be bases for a vector space V, such that

$$\bar{b}_1 = 4\bar{c}_1 + \bar{c}_2$$

and

$$\bar{b}_2 = -6\bar{c}_1 + \bar{c}_2.$$

Suppose $\bar{x} = 3\bar{b}_1 + \bar{b}_2$. Find $[\bar{x}]_{\mathcal{C}}$.

SOLUTION:

We have $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and

$$[\bar{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad [\bar{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix},$$

therefore $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$, hence

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

<u>EXAMPLE</u>:

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and $\mathcal{E}$ be the standard basis of R^3. Let also

$$[\bar{x}]_{\mathcal{E}} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}.$$

Find $[\bar{x}]_{\mathcal{B}}$.

<u>SOLUTION</u>:

We have

$$\underset{\mathcal{E}\longleftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

therefore

$$\begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} [\bar{x}]_{\mathcal{B}},$$

so

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}.$$

DEFINITION:

Let V be a vector space and B be a basis of V. The dimension of V is a number of vectors in B.

EXAMPLE:

1. Since

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \ \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is the basis for R^n, we get dim $R^n = n$.

2. Since

$$\bar{e}_1 = 1, \ \bar{e}_2 = t, \ \bar{e}_3 = t^2, \ldots, \ \bar{e}_{n+1} = t^n$$

is the basis for P^n, we get dim $P^n = n + 1$.

WARNING:

$$n\text{-dimensional space} \neq \boldsymbol{R}^n$$

EXAMPLE:

Vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

span the 3-dimensional space, since there are 3 pivots. But they do not span $\boldsymbol{R}^3$, because they have 4 coordinates.

EXAMPLE:

Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 4b + c \\ 2a - c + 3d \\ 2b - c + d \\ b + 3d \end{bmatrix} : a, b, c, d \in R \right\}$$

SOLUTION:

We have

$$\begin{bmatrix} a-4b+c \\ 2a-c+3d \\ 2b-c+2d \\ b+3d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 8 & -3 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

therefore dim $H = 4$.

THEOREM:

(a) The dimension of Nul A is the number of free variables in the equation $A\bar{x} = \bar{0}$.

(b) The dimension of Col A is the number of pivot columns in A.

EXAMPLE:

Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable x_4. Hence dim Nul A=1. Also, dim Col $A = 3$ because A has 3 pivots.

DEFINITION:

The rank of A is the dimension of the column space of A.

EXAMPLE:

Since

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have

$$\text{rank } A = 3.$$

THEOREM (THE RANK THEOREM):

(a) The dimensions of the column space and the row space of an $m \times n$ matrix A are equal.

(b) This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{ Nul } A = n.$$

<u>EXAMPLE</u>:

Let

$$\boldsymbol{A} = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Since there are 2 pivots, we have

$$\dim \text{Row } \boldsymbol{A} = \dim \text{Col } \boldsymbol{A} = 2.$$

(b) Since there are 3 free variables,

$$\dim \text{Nul } \boldsymbol{A} = 3.$$

We see that $2 + 3 = 5$ (# of columns).

DEFINITION:

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\bar{x}$ such that

$$A\bar{x} = \lambda\bar{x} \qquad (*)$$

for some scalar λ. A scalar λ is called an eigenvalue of A.

DEFINITION:

Let λ be an eigenvalue of A. The set of all solutions of $(*)$ is called the eigenspace of A corresponding to λ.

REMARK:

To find eigenvalues of A, we should solve the following characteristic equation

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then $\det(A - \lambda I) =$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

is called the <u>characteristic polynomial</u> of A and

$$\det(A - \lambda I) = 0,$$

is called the <u>characteristic equation</u> of A.

PROBLEM:

Let

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}.$$

Find all eigenvalues.

SOLUTION:

We first solve the following equation:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$(5-\lambda)(1-\lambda) = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A**.**

PROBLEM:

Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{6} \\ \mathbf{2} & \mathbf{1} & \mathbf{6} \\ \mathbf{2} & \mathbf{-1} & \mathbf{8} \end{bmatrix}.$$

An eigenvalue $\boldsymbol{\lambda}$ is 2. Find a basis for the corresponding eigenspace.

SOLUTION:

We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

hence

$$2x_1 - x_2 + 6x_3 = 0 \quad \Longrightarrow \quad x_1 = \frac{1}{2}x_2 - 3x_3$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 2$.

To find a basis for the eigenspace corresponding to $\boldsymbol{\lambda = 2}$, we note that

$$\bar{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

therefore the <u>2-dimensional eigenspace</u> corresponding to $\boldsymbol{\lambda = 2}$ is

$$H = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1,\ t_2 \in R \right\}$$

and

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the <u>basis</u> for $\boldsymbol{H}$.

DEFINITION:

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is

$$A = PDP^{-1}$$

for some invertible matrix P and some diagonal matrix D.

EXAMPLE:

Matrices

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

are similar, since

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}.$$

Also, A is diagonalizable.

THEOREM (The Diagonalization Theorem):
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case:

(a) The columns of P are n linearly independent eigenvectors of A;

(b) The diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

EXAMPLE:
One can check that $\lambda = 1, 5$ are eigenvalues of A and

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

are corresponding eigenvectors. Therefore

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}.$$

EXAMPLE:

Determine if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

SOLUTION:

We first solve the following equation:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$-\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2 = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

are eigenvalues of A, so

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

One can show that

$$\textbf{Basis for } \boldsymbol{\lambda_1 = 1}: \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\textbf{Basis for } \boldsymbol{\lambda_2 = -2}: \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

therefore

$$\boldsymbol{P} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

DEFINITION:

If $\bar{u}$ and $\bar{v}$ are vectors in R^n, then $\bar{u}^T\bar{v}$ is called the inner product (or dot product) of $\bar{u}$ and $\bar{v}$ and written as

$$\bar{u} \cdot \bar{v}$$

REMARK:

In other words, if

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

then

$$\bar{u} \cdot \bar{v} = \bar{u}^T\bar{v} = [u_1 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1v_1 + \ldots + u_nv_n.$$

EXAMPLE:

Let

$$\bar{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}.$$

Find $\bar{u} \cdot \bar{v}$.

SOLUTION:

We have

$$\bar{u} \cdot \bar{v} = 2 \cdot 3 + (-5) \cdot 2 + (-1)(-3) = -1.$$

THEOREM:

Let $\bar{u}$, $\bar{v}$, and $\bar{w}$ be vectors in R^n, and let c be a scalar. Then

(a) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

(b) $(\bar{u} + \bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$

(c) $(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v}) = \bar{u} \cdot (c\bar{v})$

(d) $\bar{u} \cdot \bar{u} \geq 0$

(d′) $\bar{u} \cdot \bar{u} = 0$ **if and only if** $\bar{u} = 0$

DEFINITION:

Let $\bar{v} = (v_1, \ldots, v_n)$ be a vector from R^n. Then the length (or norm) of $\bar{v}$ is the nonnegative scalar $\|\bar{v}\|$ defined by

$$\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

EXAMPLE:

The length of the vector $\bar{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is

$$\|\bar{u}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

PROPERTY:

Let c be a scalar. Then

$$\|c\bar{v}\| = |c|\|\bar{v}\|.$$

DEFINITION:

A vector whose length is 1 is called a unit vector.

EXAMPLE:

Let $\bar{v} = (1, -2, 2, 0)$. Find the unit vector in the same direction as $\bar{v}$.

SOLUTION:

We have

$$\|\bar{v}\| = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = \sqrt{9} = 3.$$

Put $\bar{u} = \frac{1}{\|\bar{v}\|}\bar{v}$. It is easy to show that $\bar{u}$ is the unit vector and vectors $\bar{v}$ and $\bar{u}$ have the same direction. Therefore

$$\bar{u} = \frac{1}{\|\bar{v}\|}\bar{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

DEFINITION:

Let $\bar{u}$ and $\bar{v}$ be from R^n. Then the distance between $\bar{u}$ and $\bar{v}$, written as

$$\text{dist }(\bar{u}, \bar{v}),$$

is the length of the vector $\bar{u} - \bar{v}$. That is,

$$\text{dist }(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|.$$

EXAMPLE:

Let $\bar{u} = (1, 2, 3)$ and $\bar{v} = (-1, 5, -4)$. Then

$$\bar{u} - \bar{v} = (1, 2, 3) - (-1, 5, -4) = (2, -3, 7),$$

therefore

$$\text{dist }(\bar{u}, \bar{v}) = \sqrt{2^2 + (-3)^2 + 7^2} = \sqrt{62}.$$

DEFINITION:

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal (perpendicular) if

$$\bar{u} \cdot \bar{v} = 0.$$

EXAMPLE:

Vectors $\bar{u} = (4, 12)$ and $\bar{v} = (9, -3)$ are orthogonal, since

$$\bar{u} \cdot \bar{v} = 4 \cdot 9 + 12 \cdot (-3) = 0.$$

THEOREM:

Let $\bar{u}$ and $\bar{v}$ be from R^2 or R^3 and let θ be the angle between them. Then

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}$$

EXAMPLE:

To find the angle between vectors

$$\bar{u} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix},$$

we note that $\cos\theta = \dfrac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} =$

$$\frac{5\cdot 6 + (-3)\cdot 9 + 1\cdot(-3)}{\sqrt{5^2+(-3)^2+1^2}\sqrt{6^2+9^2+(-3)^2}} = 0,$$

therefore $\theta = \dfrac{\pi}{2} = 90^\circ$.