DEFINITION:

If $\bar{u}$ and $\bar{v}$ are vectors in R^n, then $\bar{u}^T\bar{v}$ is called the inner product (or dot product) of $\bar{u}$ and $\bar{v}$ and written as

$$\bar{u} \cdot \bar{v}$$

REMARK:

In other words, if

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

then

$$\bar{u} \cdot \bar{v} = \bar{u}^T\bar{v} = [u_1 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1v_1 + \ldots + u_nv_n.$$

DEFINITION:

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal (perpendicular) if

$$\bar{u} \cdot \bar{v} = 0.$$

EXAMPLE:

Vectors $\bar{u} = (5, -3, 1)$ and $\bar{v} = (6, 9, -3)$ are orthogonal, since

$$\bar{u} \cdot \bar{v} = 5 \cdot 6 + (-3) \cdot 9 + 1 \cdot (-3) = 0.$$

ORTHOGONAL COMPLEMENTS

DEFINITION:

If a vector $\bar{z}$ is orthogonal to every vector in a subspace W of R^n, then $\bar{z}$ is said to be orthogonal to W.

EXAMPLE:

Let $\bar{z} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and

$$W = \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix} t : \ t \in R \right\}$$

be a subspace of R^2. Then $\bar{z}$ is orthogonal to every vector in W, since

$$\bar{z} \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} t \right) = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} \right) t = 0\, t = 0.$$

EXAMPLE:

Let $\bar{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and

$$W = \left\{ \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 : \ t_1, t_2 \in R \right\}$$

be a subspace of R^3. Then $\bar{z}$ is orthogonal to every vector in W, since

$$\bar{z} \cdot \left(\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2$$

$$= 0 \ t_1 + 0 \ t_2 = 0.$$

DEFINITION:

The set of all vectors $\bar{z}$ that are orthogonal to W is called the orthogonal complement of W and is denoted by

$$W^{\perp}$$

<u>EXAMPLE</u>:

Let

$$\boldsymbol{H} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} s : \ s \in \boldsymbol{R} \right\}$$

and

$$\boldsymbol{W} = \left\{ \begin{bmatrix} 6 \\ -4 \end{bmatrix} t : \ t \in \boldsymbol{R} \right\}$$

be subspaces of $\boldsymbol{R}^2$. Then every vector in $\boldsymbol{H}$ is orthogonal to every vector in $\boldsymbol{W}$, since

$$\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} s \right) \cdot \left(\begin{bmatrix} 6 \\ -4 \end{bmatrix} t \right) = \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix} \right) st = 0.$$

Moreover, one can show that there are no other vectors in $\boldsymbol{R}^2$ which are orthogonal to every vector in $\boldsymbol{W}$. Therefore $\boldsymbol{H} = \boldsymbol{W}^{\perp}$.

EXAMPLE:

Let L_1 be a line through the origin in R^2, and let L_2 be the line through the origin and perpendicular to L_1. Then each vector on L_1 is orthogonal to every vector in L_2. Moreover, one can show that there are no other vectors in R^2 which are orthogonal to every vector in L_1. Therefore

$$L_1 = L_2^{\perp}.$$

Also, for the same reason we have

$$L_2 = L_1^{\perp}.$$

EXAMPLE:

Let $\boldsymbol{H} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} s : \ s \in \boldsymbol{R} \right\}$ and

$$\boldsymbol{W} = \left\{ \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 : \ t_1, t_2 \in \boldsymbol{R} \right\}$$

be subspaces of $\boldsymbol{R}^3$. Then every vector in $\boldsymbol{H}$ is orthogonal to every vector in $\boldsymbol{W}$, since

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} s \right) \cdot \left(\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} t_1 + \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} t_2 \right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} s \ t_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix} s \ t_2$$

$$= 0 \ s \ t_1 + 0 \ s \ t_2 = 0.$$

Moreover, one can show that there are no other vectors in $\boldsymbol{R}^3$ which are orthogonal to every vector in $\boldsymbol{W}$. Therefore $\boldsymbol{H} = \boldsymbol{W}^{\perp}$.

EXAMPLE:

Let W be a plane through the origin in R^3, and let L be the line through the origin and perpendicular to W. Then each vector on L is orthogonal to every vector $\bar{z}$ in W. Moreover, one can show that there are no other vectors in R^3 which are orthogonal to every vector in W. Therefore

$$L = W^{\perp}.$$

Also, for the same reason we have

$$W = L^{\perp}.$$

THEOREM:

(a) A vector $\bar{x}$ is in $W^{\perp}$ if and only if $\bar{x}$ is orthogonal to every vector in a set that spans W.

(b) $W^{\perp}$ is a subspace of R^n.

THEOREM:

Let A be an $m \times n$ matrix. Then

$$(\text{Row } A)^{\perp} = \text{Nul } A$$

and

$$(\text{Col } A)^{\perp} = \text{Nul } A^T.$$

PROOF:

The row-column rule for computing $A\bar{x}$ shows that if $\bar{x}$ is in Nul A, then $\bar{x}$ is orthogonal to each row of A. Since the rows of A span the row space, $\bar{x}$ is orthogonal to Row A.

Conversely, if $\bar{x}$ is orthogonal to Row A, then $\bar{x}$ is certainly orthogonal to each row of A, and therefore we have $A\bar{x} = \bar{0}$.

To prove the second part of the theorem, we note that

$$(\text{Row } A^T)^{\perp} = \text{Nul } A^T \qquad (*)$$

by the first part of this theorem. On the other hand, it is easy to see that

$$\text{Row } A^T = \text{Col } A,$$

therefore

$$(\text{Row } A^T)^{\perp} = (\text{Col } A)^{\perp}. \qquad (**)$$

Combination of (*) and (**) gives the desired result. ■