PROBLEM:

Find the angle between vectors

$$\bar{u} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix}.$$

THE INNER PRODUCT

DEFINITION:

If $\bar{u}$ and $\bar{v}$ are vectors in R^n, then $\bar{u}^T\bar{v}$ is called the inner product (or dot product) of $\bar{u}$ and $\bar{v}$ and written as

$$\bar{u} \cdot \bar{v}$$

REMARK:

In other words, if

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

then

$$\bar{u} \cdot \bar{v} = \bar{u}^T\bar{v} = [u_1 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1v_1 + \ldots + u_nv_n.$$

EXAMPLE:

Let

$$\bar{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}.$$

Find $\bar{u} \cdot \bar{v}$.

SOLUTION:

We have

$$\bar{u} \cdot \bar{v} = 2 \cdot 3 + (-5) \cdot 2 + (-1)(-3) = -1.$$

THEOREM:

Let $\bar{u}$, $\bar{v}$, and $\bar{w}$ be vectors in R^n, and let c be a scalar. Then

(a) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

(b) $(\bar{u} + \bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$

(c) $(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v}) = \bar{u} \cdot (c\bar{v})$

(d) $\bar{u} \cdot \bar{u} \geq 0$

(d′) $\bar{u} \cdot \bar{u} = 0$ **if and only if** $\bar{u} = 0$

THE LENGTH OF A VECTOR

DEFINITION:

Let $\bar{v} = (v_1, \ldots, v_n)$ be a vector from R^n. Then the length (or norm) of $\bar{v}$ is the nonnegative scalar $\|\bar{v}\|$ defined by

$$\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

EXAMPLE:

The length of the vector

$$\bar{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

is

$$\|\bar{u}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

PROPERTY:

Let c be a scalar. Then

$$\|c\bar{v}\| = |c|\|\bar{v}\|.$$

PROOF 1:

We have

$$\begin{aligned} \|c\bar{v}\| &= \sqrt{(cv_1)^2 + \ldots + (cv_n)^2} \\ &= \sqrt{c^2(v_1^2 + \ldots + v_n^2)} \\ &= |c|\sqrt{v_1^2 + \ldots + v_n^2} \\ &= |c|\|\bar{v}\|. \end{aligned}$$

PROOF 2:

We have

$$\|c\bar{v}\|^2 = (c\bar{v}) \cdot (c\bar{v}) = c^2\bar{v} \cdot \bar{v} = c^2\|\bar{v}\|^2.$$

DEFINITION:

A vector whose length is 1 is called a unit vector.

PROBLEM:

Let $\bar{v} = (1, -2, 2, 0)$. Find:

(a) The length of $\bar{v}$;

(b) The unit vector in the same direction as $\bar{v}$.

SOLUTION:

(a) We have

$$\|\bar{v}\| = \sqrt{1^2 + (-2)^2 + 2^2 + 0^2} = \sqrt{9} = 3.$$

(b) Put

$$\bar{u} = \frac{1}{\|\bar{v}\|}\bar{v}.$$

Note that vectors $\bar{v}$ and $\bar{u}$ have the same direction. Moreover, since

$$\|\bar{u}\| = \left\|\frac{1}{\|\bar{v}\|}\bar{v}\right\| = \frac{1}{\|\bar{v}\|}\|\bar{v}\| = 1,$$

it follows that $\bar{u}$ is the unit vector. Finally, we have

$$\|\bar{u}\| = \frac{1}{\|\bar{v}\|}\bar{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}.$$

DISTANCE IN R^n

DEFINITION:

Let $\bar{u}$ and $\bar{v}$ be from R^n. Then the distance between $\bar{u}$ and $\bar{v}$, written as

$$\text{dist }(\bar{u}, \bar{v}),$$

is the length of the vector $\bar{u} - \bar{v}$. That is,

$$\text{dist }(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|.$$

EXAMPLE:

Let

$$\bar{u} = (1, 2, 3) \quad \text{and} \quad \bar{v} = (-1, 5, -4).$$

Find the distance between $\bar{u}$ and $\bar{v}$.

SOLUTION:

Step 1: We have

$$\bar{u} - \bar{v} = (1, 2, 3) - (-1, 5, -4) = (2, -3, 7).$$

Step 2: By the Definiton above, we get

$$\text{dist}\ (\bar{u}, \bar{v}) = \sqrt{2^2 + (-3)^2 + 7^2} = \sqrt{62}.$$

ORTHOGONAL VECTORS

DEFINITION:

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal (perpendicular) if

$$\bar{u} \cdot \bar{v} = 0.$$

EXAMPLE:

Vectors $\bar{u} = (4, 12)$ and $\bar{v} = (9, -3)$ are orthogonal, since

$$\bar{u} \cdot \bar{v} = 4 \cdot 9 + 12 \cdot (-3) = 0.$$

THEOREM (The Pythagorean Theorem):

Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal if and only if

$$\|\bar{u} + \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2.$$

ANGLES IN R^2 AND R^3

THEOREM:

Let $\bar{u}$ and $\bar{v}$ be from R^2 or R^3 and let θ be the angle between them. Then

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|}$$

EXAMPLE:

Find the angle between vectors

$$\bar{u} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} 6 \\ 9 \\ -3 \end{bmatrix}.$$

SOLUTION:

We have $\cos\theta = \dfrac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} =$

$$\frac{5\cdot 6+(-3)\cdot 9+1\cdot(-3)}{\sqrt{5^2+(-3)^2+1^2}\sqrt{6^2+9^2+(-3)^2}}=0,$$

therefore

$$\theta=\frac{\pi}{2}=90^\circ.$$