PROBLEM:

Let $T : R^2 \longrightarrow R^2$ be a linear operator such that

$$T(\bar{x}) = A\bar{x}, \quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

Find all nonzero vectors $\bar{x} \in R^2$ and all scalars λ such that

$$T(\bar{x}) = \lambda\bar{x}.$$

SOLUTION:

Suppose there is a vector $\bar{x} \in R^2$ and a scalar λ such that

$$T(\bar{x}) = \lambda\bar{x}.$$

Since $T(\bar{x}) = A\bar{x}$, we rewrite this as

$$A\bar{x} = \lambda\bar{x},$$

hence

$$A\bar{x} - \lambda\bar{x} = \bar{0},$$

so

$$(A - \lambda I)\bar{x} = \bar{0}.$$

THEOREM:

Let B be a square $n \times n$ matrix. Then the equation

$$B\bar{x} = \bar{0}$$

has a nontrivial solution if and only if

$$\det B = 0.$$

By the Theorem above,

$$(A - \lambda I)\bar{x} = \bar{0}.$$

has a nontrivial solution if and only if

$$\det(A - \lambda I) = 0. \qquad (*)$$

Note that

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix},$$

therefore we can rewrite $(*)$ as

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$\lambda^2 - 3\lambda + 2 = 0.$$

Solving this quadratic equation, we get

$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

(a) Let $\lambda = 1$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Let $\lambda = 2$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - 2x_2 = 0 \quad \Longrightarrow \quad x_1 = 2x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Conclusion:

The equation

$$T(\bar{x}) = \lambda \bar{x}$$

has a nonzero solution $\bar{x} \in R^2$ if and only if

$$\lambda = 1 \text{ and } \bar{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \text{ and } \bar{x} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where t is any real number.

DEFINITION:

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\bar{x}$ such that

$$A\bar{x} = \lambda\bar{x} \qquad (*)$$

for some scalar λ. A scalar λ is called an eigenvalue of A.

DEFINITION:

Let λ be an eigenvalue of A. The set of all solutions of $(*)$ is called the eigenspace of A corresponding to λ.

REMARK:

To find eigenvalues of A, we should solve the following characteristic equation

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix.

Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) =$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

is called the <u>characteristic polynomial</u> of $\boldsymbol{A}$ and

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0,$$

is called the <u>characteristic equation</u> of $\boldsymbol{A}$.

PROBLEM:

Let

$$A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}.$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

SOLUTION:

We first solve the following equation:

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$(5-\lambda)(1-\lambda) = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A.

(a) Let $\boldsymbol{\lambda = 1}$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 = 0.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 1$.

The 1-dimensional eigenspace corresponding to $\lambda = 1$ is

$$H = \left\{ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the basis for H.

(b) Let $\boldsymbol{\lambda} = 5$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - 2x_2 = 0 \quad \Longrightarrow \quad x_1 = 2x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 5$.

The 1-dimensional eigenspace corresponding to $\lambda = 5$ is

$$H = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the basis for H.

PROBLEM:

Let

$$\boldsymbol{A} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}.$$

An eigenvalue $\boldsymbol{\lambda}$ is 2. Find a basis for the corresponding eigenspace.

SOLUTION:

We use row operations:

$$\begin{bmatrix} 4-\lambda & -1 & 6 & 0 \\ 2 & 1-\lambda & 6 & 0 \\ 2 & -1 & 8-\lambda & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

hence

$$2x_1 - x_2 + 6x_3 = 0 \quad \Longrightarrow \quad x_1 = \frac{1}{2}x_2 - 3x_3$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

is the eigenvector of A, corresponding to $\lambda = 2$.

To find a basis for the eigenspace corresponding to $\lambda = 2$, we note that

$$\bar{x} = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

therefore the <u>2-dimensional eigenspace</u> corresponding to $\lambda = 2$ is

$$H = \left\{ t_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : t_1,\ t_2 \in R \right\}$$

and

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the <u>basis</u> for H.

PROBLEM:

Let

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Find a formula for A^k and D^k.

SOLUTION:

(a) We have

$$D^2 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 1^k & 0 \\ 0 & 5^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix}$$

(b) We have

$$\boldsymbol{A}^2 = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix}$$

$$\boldsymbol{A}^3 = \begin{bmatrix} 37 & 24 \\ -18 & -11 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix}$$

$$\boldsymbol{A}^4 = \begin{bmatrix} 187 & 124 \\ -93 & -61 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix}$$

$$\boldsymbol{A}^5 = \begin{bmatrix} 937 & 624 \\ -468 & -311 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 4687 & 3124 \\ -2343 & -1561 \end{bmatrix}$$

Let

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$$

One can check that

$$A = PDP^{-1}.$$

We have

$$\begin{aligned} A^2 &= PDP^{-1}PDP^{-1} \\ &= PD\underbrace{(P^{-1}P)}_{I}DP^{-1} \\ &= PDDP^{-1} = PD^2P^{-1} \end{aligned}$$

Similarly,

$$\begin{aligned} A^3 &= PD^3P^{-1} \\ A^4 &= PD^4P^{-1} \\ A^5 &= PD^5P^{-1} \end{aligned}$$

In general,

$$A^k = PD^kP^{-1} = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 \\ -3/4 & -1/2 \end{bmatrix}$$

DEFINITION:

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is

$$A = PDP^{-1}$$

for some invertible matrix P and some diagonal matrix D.

EXAMPLE:

Matrices

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

are similar, since

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}.$$

Also, A is diagonalizable.

THEOREM (The Diagonalization Theorem):
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case:

(a) The columns of P are n linearly independent eigenvectors of A;

(b) The diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

EXAMPLE:
One can check that $\lambda = 1, 5$ are eigenvalues of A and

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

are corresponding eigenvectors. Therefore

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \text{ and } P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}.$$

EXAMPLE:

Determine if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

SOLUTION:

We first solve the following equation:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$-\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2 = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

are eigenvalues of A, so

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

One can show that

$$\textbf{Basis for } \boldsymbol{\lambda_1 = 1}: \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\textbf{Basis for } \boldsymbol{\lambda_2 = -2}: \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

therefore

$$\boldsymbol{P} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$