PROBLEM:

Let $T : R^2 \longrightarrow R^2$ be a linear operator such that

$$T(\bar{x}) = A\bar{x}, \quad A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}.$$

Let also

$$\bar{x}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \bar{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find $T(\bar{x}_1)$, $T(\bar{x}_2)$, and $T(\bar{x}_3)$.

SOLUTION:

We have

$$T(\bar{x}_1) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$T(\bar{x}_2) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$$

$$T(\bar{x}_3) = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$$

PROBLEM:

Let $T : R^2 \longrightarrow R^2$ be a linear operator such that

$$T(\bar{x}) = A\bar{x}, \quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

(a) Find a nonzero vector $\bar{x} \in R^2$ such that

$$T(\bar{x}) = \bar{x}.$$

(b) Find a nonzero vector $\bar{x} \in R^2$ such that

$$T(\bar{x}) = 2\bar{x}$$

(c) Find all nonzero vectors $\bar{x} \in R^2$ and all scalars λ such that

$$T(\bar{x}) = \lambda\bar{x}.$$

SOLUTION:

Suppose there is a vector $\bar{x} \in R^2$ and a scalar λ such that

$$T(\bar{x}) = \lambda \bar{x}.$$

Since $T(\bar{x}) = A\bar{x}$, we rewrite this as

$$A\bar{x} = \lambda \bar{x},$$

hence

$$A\bar{x} - \lambda \bar{x} = \bar{0},$$

so

$$(A - \lambda I)\bar{x} = \bar{0}. \qquad (*)$$

So, we should find such λ that $(*)$ has a nontrivial solution.

<u>THEOREM</u>:

Let A be a square $n \times n$ matrix. Then the following statements are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) The equation $A\bar{x} = \bar{0}$ has only the trivial solution.

(d) The columns of A form a linearly independent set.

(e) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n.

(f) The columns of A span R^n.

(g) A^T is an invertible matrix.

(h) A has n pivot positions.

COROLLARY:

Let A be a square $n \times n$ matrix. Then the equation

$$A\bar{x} = \bar{0}$$

has a nontrivial solution if and only if

$$\det A = 0.$$

Suppose there is a vector $\bar{x} \in R^2$ and a scalar λ such that

$$A\bar{x} = \lambda\bar{x},$$

hence

$$(A - \lambda I)\bar{x} = \bar{0}. \qquad (*)$$

By the Corollary above, $(*)$ has a non-trivial solution if and only if

$$\det(A - \lambda I) = 0. \qquad (**)$$

Note that

$$A - \lambda I = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{bmatrix},$$

therefore we can rewrite $(**)$ as

$$\begin{vmatrix} 3 - \lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0.$$

So, the equation

$$A\bar{x} = \lambda\bar{x},$$

has a nonzero solution $\bar{x} \in R^2$ if and only if λ satisfies the equation

$$\begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$-(3-\lambda)\lambda + 2 = 0,$$

hence

$$\lambda^2 - 3\lambda + 2 = 0.$$

Solving this quadratic equation, we get

$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

Conclusion: The equation

$$A\bar{x} = \lambda\bar{x},$$

has a nonzero solution $\bar{x} \in R^2$ if and only if

$$\lambda = 1 \text{ or } 2.$$

(a) Let $\boldsymbol{\lambda} = 1$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Let $\lambda = 2$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - 2x_2 = 0 \quad \Longrightarrow \quad x_1 = 2x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Conclusion:

The equation

$$A\bar{x} = \lambda\bar{x},$$

has a nonzero solution $\bar{x} \in R^2$ if and only if

$$\lambda = 1 \text{ and } \bar{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \text{ and } \bar{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where x_2 is any real number.

DEFINITION:

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\bar{x}$ such that

$$A\bar{x} = \lambda\bar{x}$$

for some scalar λ. A scalar λ is called an eigenvalue of A.

EXAMPLE:

Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

Then $\lambda = 1$ and $\lambda = 2$ are eigenvalues of A and

$$t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are eigenvectors of A, where t is any real number.

DEFINITION:

Let A be an $n \times n$ matrix and let λ be an eigenvalue. The set of all solutions of the equation

$$(A - \lambda I)\bar{x} = \bar{0}$$

is called the eigenspace of A corresponding to λ.

EXAMPLE:

Let

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of A corresponding to $\lambda = 1$;

$$\left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

is the eigenspace of A corresponding to $\lambda = 2$;

PROBLEM:

Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}.$$

Find all eigenvalues, eigenvectors and bases for the corresponding eigenspaces.

SOLUTION:

We first solve the following equation:

$$\begin{vmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$(5-\lambda)(1-\lambda) = 0,$$

hence

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

are eigenvalues of A.

(a) Let $\lambda = 1$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 = 0.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the <u>eigenvector</u> of A, corresponding to $\lambda = 1$.

The <u>1-dimensional eigenspace</u> corresponding to $\lambda = 1$ is

$$H = \left\{ t \begin{bmatrix} 0 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the <u>basis</u> for H.

(b) Let $\lambda = 5$. To solve the homogeneous system

$$(A - \lambda I)\bar{x} = \bar{0},$$

we use row operations:

$$\begin{bmatrix} 5-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$x_1 - 2x_2 = 0 \quad \Longrightarrow \quad x_1 = 2x_2.$$

We get

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is the <u>eigenvector</u> of A, corresponding to $\lambda = 5$.

The <u>1-dimensional eigenspace</u> corresponding to $\lambda = 5$ is

$$H = \left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \text{ is any real number} \right\}$$

and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the <u>basis</u> for H.