

PROBLEM:

Consider the vector space

$$R^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in R \right\}$$

and 2 bases of R^3 :

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Find coordinates of the vector

$$\bar{x} = (-4, 3, -5)$$

in $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and in $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

DEFINITION:

Suppose $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ is a basis for a vector space V and \bar{x} is in V . The coordinates of \bar{x} relative to the basis \mathcal{B} are the weights c_1, \dots, c_n such that

$$\bar{x} = c_1\bar{b}_1 + \dots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$$

SOLUTION:

(a) Let

$$\mathcal{B}_1 = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}.$$

To find coordinates of the vector

$$\bar{x} = (-4, 3, -5)$$

relative to \mathcal{B}_1 , we consider the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix},$$

therefore

$$[\bar{x}]_{\mathcal{B}_1} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}.$$

(b) Let

$$\mathcal{B}_2 = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}.$$

To find coordinates of the vector

$$\bar{x} = (-4, 3, -5)$$

relative to \mathcal{B}_2 , we consider the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

therefore

$$[\bar{x}]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}.$$

CONCLUSION:

The vector

$$\bar{x} = (-4, 3, -5)$$

has two different coordinates in two different bases:

$$[\bar{x}]_{\mathcal{B}_1} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} \quad \text{and} \quad [\bar{x}]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix},$$

where

$$\mathcal{B}_1 = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$$

and

$$\mathcal{B}_2 = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}.$$

THEOREM:

Let $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ and $\mathcal{C} = \{\bar{c}_1, \dots, \bar{c}_n\}$ be bases of a vector space V . Then there is a unique matrix ${}_{\mathcal{C}}P_{\mathcal{B}}$ such that

$$[\bar{x}]_{\mathcal{C}} = {}_{\mathcal{C}}P_{\mathcal{B}} [\bar{x}]_{\mathcal{B}},$$

where

$${}_{\mathcal{C}}P_{\mathcal{B}} = [[\bar{b}_1]_{\mathcal{C}} \quad [\bar{b}_2]_{\mathcal{C}} \quad \dots \quad [\bar{b}_n]_{\mathcal{C}}].$$

EXAMPLE:

Let $\bar{x} = (-4, 3, -5)$, $C = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$
and $B = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, then

$$[\bar{x}]_C = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} \quad [\bar{x}]_B = \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix},$$

$$[\bar{v}_1]_C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [\bar{v}_2]_C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [\bar{v}_3]_C = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

therefore

$${}_{C \leftarrow B} P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and in fact

$$\begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 1 \end{bmatrix}$$

REMARK:

One can show that

$$\left(\begin{matrix} P \\ \mathcal{C} \longleftarrow \mathcal{B} \end{matrix} \right)^{-1} = \begin{matrix} P \\ \mathcal{B} \longleftarrow \mathcal{C} \end{matrix}$$

EXAMPLE:

We have $\begin{matrix} P \\ \mathcal{C} \longleftarrow \mathcal{B} \end{matrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} [\bar{x}]_{\mathcal{B}}$$

therefore

$$[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} [\bar{x}]_{\mathcal{C}},$$

so

$$\begin{matrix} P \\ \mathcal{B} \longleftarrow \mathcal{C} \end{matrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \left(\begin{matrix} P \\ \mathcal{C} \longleftarrow \mathcal{B} \end{matrix} \right)^{-1}.$$

PROBLEM:

Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$ be bases for a vector space V , such that

$$\bar{b}_1 = 4\bar{c}_1 + \bar{c}_2$$

and

$$\bar{b}_2 = -6\bar{c}_1 + \bar{c}_2.$$

Suppose

$$\bar{x} = 3\bar{b}_1 + \bar{b}_2.$$

Find $[\bar{x}]_{\mathcal{C}}$.

SOLUTION:

We have $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and

$$[\bar{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad [\bar{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix},$$

therefore

$${}_{\mathcal{C}}P_{\leftarrow \mathcal{B}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix},$$

hence

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

PROBLEM:

Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$ be bases for a vector space V , such that

$$\bar{b}_1 = 6\bar{c}_1 - 2\bar{c}_2$$

and

$$\bar{b}_2 = 9\bar{c}_1 - 4\bar{c}_2.$$

Suppose

$$\bar{x} = -3\bar{b}_1 + 2\bar{b}_2.$$

Find $[\bar{x}]_{\mathcal{C}}$.

SOLUTION:

We have $[\bar{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and

$$[\bar{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad [\bar{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix},$$

therefore

$${}_{\mathcal{C}}P_{\leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix},$$

hence

$$[\bar{x}]_{\mathcal{C}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

PROBLEM:

Let $\bar{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\bar{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\bar{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\bar{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for R^2 given by $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$.

(a) Find the change-of-coordinates matrix from \mathcal{C} to \mathcal{B} .

(b) Find the change-of-coordinates matrix from \mathcal{B} to \mathcal{C} .