DEFINITION:

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following 10 axioms (or rules):

1. **The sum of $\bar{u}$ and $\bar{v}$, denoted by $\bar{u} + \bar{v}$, is in V.**
2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.
3. $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$.
4. **There is a zero vector $\bar{0}$ in V such that $\bar{u} + \bar{0} = \bar{u}$.**
5. **For each $\bar{u}$ in V, there is a vector $-\bar{u}$ in V such that $\bar{u} + (-\bar{u}) = \bar{0}$.**
6. **The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.**
7. $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$.
8. $(c + d)\bar{u} = c\bar{u} + d\bar{u}$.
9. $c(d\bar{u}) = (cd)\bar{u}$.
10. $1 \cdot \bar{u} = \bar{u}$.

These axioms must hold for all vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ in V and all scalars c and d.

EXAMPLE:

1. $$R^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in R \right\}$$

2. **The set P_n of polynomials of degree at most n:**

$$\bar{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable t are real numbers.

3. **The set of all real-valued functions defined on R.**

DEFINITION:

A subspace of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\bar{u}$ and $\bar{v}$ in H, the sum $\bar{u} + \bar{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H.

EXAMPLE:

The set consisting of only the zero vector $\bar{0}$ in a vector space V is a subspace of V, called the zero subspace and written as $\{\bar{0}\}$.

WARNING:

R^2 is not a subspace of R^3, because R^2 is not a subset of R^3.

THEOREM:

If $\bar{v}_1, \ldots, \bar{v}_p$ are in a vector space V, then Span $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a subspace of V.

EXAMPLE:

Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

By the Theorem above

$$\text{Span}\{\bar{v}_1, \bar{v}_2\}$$

is a subspace of R^3.

EXAMPLE:

The set

$$H = \left\{ \begin{bmatrix} 4a - b \\ 2b \\ a - 2b \\ a - b \end{bmatrix} : a, b \in R \right\}$$

is a subspace of R^4, because

$$\begin{bmatrix} 4a - b \\ 2b \\ a - 2b \\ a - b \end{bmatrix} = a \underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\bar{v}_1} + b \underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}}_{\bar{v}_2}.$$

DEFINITION:

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation

$$A\bar{x} = \bar{0}.$$

DEFINITION:

Let A be an $m \times n$ matrix. The row space is the set of all linear combinations of the row vectors of A.

DEFINITION:

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A.

EXAMPLE:

Find a spanning set for the column space, row space, and null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

SOLUTION:

(a) Obviously, columns of A, i.e.

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \quad \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix}$$

form the spanning set for Col A.

(b) Obviously, rows of A, i.e.

$$(-3, \quad 6, -1, 1, -7)$$
$$(1, -2, \quad 2, 3, -1)$$
$$(2, -4, \quad 5, 8, -4)$$

form the spanning set for the row space of A.

(c) To find a spanning set for Nul A, we find the general solution of $A\bar{x} = \bar{0}$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0, \end{cases}$$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$

$$= x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}},$$

so Nul A =Span $\{\bar{u}, \bar{v}, \bar{w}\}$.

DEFINITION:

Let H be a subspace of a vector space V. A set of vectors

$$B = \{\bar{b}_1, \ldots, \bar{b}_p\}$$

in V is a basis for H if

(a) B is a linearly independent set;

(b) $H = \text{Span}\,\{\bar{b}_1, \ldots, \bar{b}_p\}$.

EXAMPLE:

The set of vectors

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

form a basis (so-called, standard basis) for R^n.

THEOREM:

The set of vectors $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a basis of R^n if and only if $n = p$ and the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has exactly n pivot positions.

EXAMPLE:

The set of vectors

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

form a basis for R^3, since

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

and we have 3 vectors and 3 pivots.

EXAMPLE:

Find bases for the row space, the column space, and the null space of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Since the first two rows have pivots, they form a basis for the row space of A.

(b) Since the first two columns have pivots, they form a basis for Col A.

(c) Finally, for Nul A we need the reduced echelon form. We have:

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding system is

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

so $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is the basis for Nul A.

DEFINITION:

Suppose $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis for a vector space V and $\bar{x}$ is in V. The coordinates of $\bar{x}$ relative to the basis B are the weights $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_B = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

THEOREM:

Let $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis for a vector space V. Then for each $\bar{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

EXAMPLE:

Let

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Find coordinates of $\bar{x}$ in $\{\bar{b}_1, \ \bar{b}_2\}$.

SOLUTION:

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix},$$

therefore

$$c_1 = -2 \quad \text{and} \quad c_2 = 3,$$

so

$$[\bar{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

DEFINITION:

Let V be a vector space and B be a basis of V. The dimension of V is a number of vectors in B.

EXAMPLE:

Since

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \ \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is the basis for R^n, we get dim $R^n = n$.

EXAMPLE:

Subspaces of R^3 can be classified by dimension:

0-dimensional subspaces: Only the zero subspace.

1-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.

2-dimensional subspaces: Any subspace spanned by 2 linearly independent vectors (= not parallel). Such subspaces are planes through the origin.

3-dimensional subspaces: Only R^3 itself. Any 3 linearly independent vectors in R^3 (= not in the same plane) span all of R^3.

WARNING:

$$n\text{-dimensional space} \neq \boldsymbol{R}^n$$

EXAMPLE:

Vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

span the 3-dimensional space, since there are 3 pivots. But they do not span $\boldsymbol{R}^3$, because they have 4 coordinates.

THEOREM:

Let V be a p-dimensional vector space, $p \geq 1$. Then

(a) Any linearly independent set of exactly p elements in V is automatically a basis for V.

(b) Any set of exactly p elements that spans V is automatically a basis for V.

<u>THEOREM</u>:

(a) The dimension of Nul A is the number of free variables in the equation $A\bar{x} = \bar{0}$.

(b) The dimension of Col A is the number of pivot columns in A.

<u>EXAMPLE</u>:

Find the dimensions of the null space and the column space of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable x_4. Hence dim Nul A=1. Also, dim Col $A = 3$ because A has 3 pivots.

DEFINITION:

The rank of A is the dimension of the column space of A.

EXAMPLE:

Since

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have

$$\text{rank } A = 3.$$

THEOREM (THE RANK THEOREM):

(a) The dimensions of the column space and the row space of an $m \times n$ matrix A are equal.

(b) This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

<u>EXAMPLE</u>:

Let

$$A = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Since there are 2 pivots, we have

$$\dim \text{Row } A = \dim \text{Col } A = 2.$$

(b) Since there are 3 free variables,

$$\dim \text{Nul } A = 3.$$

We see that $2 + 3 = 5$ (# of columns).

<u>EXAMPLE</u>:

(a) If A is a 5×11 matrix with a 7-dimensional null space, what is the rank of A.

(b) Could a 5×11 matrix have a 5-dimensional null space?

SOLUTION:

(a) Since A has 11 columns, by the Theorem above we have

$$(\text{rank } A) + 7 = 11,$$

and hence rank $A = 4$

(b) No. If a 5×11 matrix had a 5-dimensional null space, it would have to have rank 6 by the Theorem above. But A has only 5 rows, therefore rank cannot exceed 5.

THEOREM:

Let A be a square $n \times n$ matrix. Then the following statements are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\bar{x} = \bar{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n.

(g) The columns of A span R^n.

(h) A^T is an invertible matrix.

(i) The columns of A form a basis of R^n.

(j) Col $A = R^n$

(k) dim Col $A = n$

(l) rank $A = n$

(m) Nul $A = \{\bar{0}\}$

(n) dim Nul $A = 0$