DEFINITION:

Let V be a vector space and B be a basis of V. The dimension of V is a number of vectors in B.

EXAMPLE:

Since

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \ \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is the basis for R^n, we get dim $R^n = n$.

EXAMPLE:

Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 4b + c \\ 2a - c + 3d \\ 2b - c + d \\ b + 3d \end{bmatrix} : a, b, c, d \in R \right\}$$

SOLUTION:

We have

$$\begin{bmatrix} a-4b+c \\ 2a-c+3d \\ 2b-c+2d \\ b+3d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

Using elementary row operations, we get

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 2 & 0 & -1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 8 & -3 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

therefore $\dim H = 4$.

EXAMPLE:

Subspaces of R^3 can be classified by dimension:

0-dimensional subspaces: Only the zero subspace.

1-dimensional subspaces: Any subspace spanned by a single nonzero vector. Such subspaces are lines through the origin.

2-dimensional subspaces: Any subspace spanned by 2 linearly independent vectors (= not parallel). Such subspaces are planes through the origin.

3-dimensional subspaces: Only R^3 itself. Any 3 linearly independent vectors in R^3 (= not in the same plane) span all of R^3.

<u>THEOREM</u>:

Let V be a p-dimensional vector space, $p \geq 1$. Then

(a) Any linearly independent set of exactly p elements in V is automatically a basis for V.

(b) Any set of exactly p elements that spans V is automatically a basis for V.

<u>THEOREM</u>:

(a) The dimension of Nul A is the number of free variables in the equation $A\bar{x} = \bar{0}$.

(b) The dimension of Col A is the number of pivot columns in A.

EXAMPLE:

Find the dimensions of the null space and the column space of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -2 \\ 4 & 4 & -1 & -4 \\ 7 & 6 & 2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable x_4. Hence dim Nul A=1. Also, dim Col $A = 3$ because A has 3 pivots.

DEFINITION:

Let A be an $m \times n$ matrix. The row space is the set of all linear combinations of the row vectors of A.

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The row space of A is the subspace of R^4 spanned by

$$\bar{v}_1 = (1, 2, 3, 4)$$

$$\bar{v}_2 = (5, 6, 7, 8)$$

$$\bar{v}_3 = (0, 0, 1, 2)$$

THEOREM:

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

EXAMPLE:

Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix}$$

SOLUTION:

Using elementary row operations, we get

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 2 & 0 & 3 \\ 0 & 9 & -3 & -1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) By the Theorem above, the first two rows of the second matrix form a basis for the row space of A.

(b) Since pivots are in columns 1 and 2, the first two columns of A form a basis for Col A.

(c) Finally, for Nul A we need the reduced echelon form. We have:

$$\begin{bmatrix} -1 & 4 & -2 & 0 & -3 \\ 2 & 1 & 1 & -1 & 0 \\ 0 & 9 & -3 & -1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{9} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

therefore the corresponding system is

$$\begin{cases} x_1 + \frac{2}{3}x_3 - \frac{4}{9}x_4 + \frac{1}{3}x_5 = 0 \\ x_2 - \frac{1}{3}x_3 - \frac{1}{9}x_4 - \frac{2}{3}x_5 = 0 \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 + \frac{4}{9}x_4 - \frac{1}{3}x_5 \\ \frac{1}{3}x_3 + \frac{1}{9}x_4 + \frac{2}{3}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} \frac{4}{9} \\ \frac{1}{9} \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2} + x_5 \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_3}$$

so $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is the basis for Nul A.