DEFINITION:

Let H be a subspace of a vector space V. A set of vectors

$$B = \{\bar{b}_1, \ldots, \bar{b}_p\}$$

in V is a basis for H if

(a) B is a linearly independent set;

(b) $H = \text{Span}\ \{\bar{b}_1, \ldots, \bar{b}_p\}$.

REMARK:

In other words, a basis for H is a minimal number of vectors which span H.

EXAMPLE:

Let

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis for R^n, because

(a) they are linearly independent, since

$$(\# \text{ of columns}) = (\# \text{ or pivots})$$

(b) they span R^n, since

there are n pivots.

DEFINITION:

The set $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is called the standard basis for R^n.

THEOREM:

The set of vectors $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a basis of R^n if and only if $n = p$ and the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has exactly n pivot positions.

PROBLEM:

Let

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Determine if $\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$ is a basis for R^3.

SOLUTION:

We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since we have 3 vectors and 3 pivots, $\{\bar{v}_1,\ \bar{v}_2,\ \bar{v}_3\}$ is a basis for R^3.

THEOREM:

The pivot columns of a matrix A form a basis for Col A.

PROBLEM:

Let

$$\bar{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Find a basis for Col $[\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3]$.

SOLUTION:

We have

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the first and the second columns are pivot columns, $\{\bar{v}_1,\ \bar{v}_2\}$ is a basis for Col $[\bar{v}_1\ \bar{v}_2\ \bar{v}_3]$.

PROBLEM:

It can be shown that the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the bases for Col A and Nul A.

SOLUTION:

(a) By the Theorem above, $\{\bar{v}_1,\ \bar{v}_3,\ \bar{v}_5\}$ is a basis for Col A.

(b) To find the basis for Nul A, we consider a system

$$\begin{cases} x_1 + 4x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \\ x_5 = 0. \end{cases}$$

Write the general solution in the parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{v}_1} + x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2}$$

so $\{\bar{v}_1, \bar{v}_2\}$ is the basis for Nul A.

TEST 1:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are linearly independent if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has p pivots.

TEST 2:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ span R^n if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has n pivots.

TEST 3:

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ form a basis of R^n if and only if the matrix $A = [\bar{v}_1 \ldots \bar{v}_p]$ has n pivots and $p = n$.

EXAMPLE:

(a) $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix}$

(b) $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{bmatrix}$

(c) $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{8} \\ \mathbf{10} \\ \mathbf{12} \end{bmatrix}$

DEFINITION:

Suppose $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis for a vector space V and $\bar{x}$ is in V. The coordinates of $\bar{x}$ relative to the basis B are the weights $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

NOTATION:

$$[\bar{x}]_B = \begin{bmatrix} c_1 \\ \ldots \\ c_n \end{bmatrix}$$

THEOREM:

Let $B = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be a basis for a vector space V. Then for each $\bar{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\bar{x} = c_1\bar{b}_1 + \ldots + c_n\bar{b}_n.$$

<u>EXAMPLE</u>:

Let

$$\bar{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \bar{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Find coordinates of $\bar{x}$ in $\{\bar{b}_1, \ \bar{b}_2\}$.

SOLUTION:

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix},$$

therefore

$$c_1 = -2 \quad \text{and} \quad c_2 = 3,$$

so

$$[\bar{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$