DEFINITION:

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following 10 axioms (or rules):

1. The sum of $\bar{u}$ and $\bar{v}$, denoted by $\bar{u} + \bar{v}$, is in V.

2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.

3. $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$.

4. There is a zero vector $\bar{0}$ in V such that $\bar{u} + \bar{0} = \bar{u}$.

5. For each $\bar{u}$ in V, there is a vector $-\bar{u}$ in V such that $\bar{u} + (-\bar{u}) = \bar{0}$.

6. The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.

7. $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$.

8. $(c + d)\bar{u} = c\bar{u} + d\bar{u}$.

9. $c(d\bar{u}) = (cd)\bar{u}$.

10. $1 \cdot \bar{u} = \bar{u}$.

These axioms must hold for all vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ in V and all scalars c and d.

EXAMPLE:

1. $$R^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in R \right\}$$

2. **The set P_n of polynomials of degree at most n:**

$$\bar{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable t are real numbers.

3. **The set of all real-valued functions defined on R.**

DEFINITION:

A subspace of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.

2. H is closed under vector addition. That is, for each $\bar{u}$ and $\bar{v}$ in H, the sum $\bar{u}+\bar{v}$ is in H.

3. H is closed under multiplication by scalars. That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H.

EXAMPLE:

The set consisting of only the zero vector $\bar{0}$ in a vector space V is a subspace of V, called the zero subspace and written as $\{\bar{0}\}$.

WARNING:

R^2 **is not a subspace of** R^3**, because** R^2 **is not a subset of** R^3**.**

EXAMPLE:

The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real numbers} \right\}$$

is a subspace of R^3**.**

THEOREM:

If $\bar{v}_1, \ldots, \bar{v}_p$ are in a vector space V, then Span $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a subspace of V.

EXAMPLE:

Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

By the Theorem above

$$\text{Span}\{\bar{v}_1, \bar{v}_2\}$$

is a subspace of R^3.

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} 4a - b \\ 2b \\ a - 2b \\ a - b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is a subspace of R^4.

SOLUTION:

We have

$$\begin{bmatrix} 4a-b \\ 2b \\ a-2b \\ a-b \end{bmatrix} = a\underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\bar{v}_1} + b\underbrace{\begin{bmatrix} -1 \\ 2 \\ -2 \\ -1 \end{bmatrix}}_{\bar{v}_2}$$

We see that

$$H = \text{Span}\{\bar{v}_1, \bar{v}_2\}$$

therefore H is a subspace of R^4 by the Theorem above.

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$$

where a, b and c are arbitrary scalars. Find a set S of vectors that spans H or show that H is not a vector space.

<u>SOLUTION:</u>

We have

$$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\bar{v}_1} + b\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}_2} + c\underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_3}$$

and we see that H is a vector space and

$$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$$

spans H.

EXAMPLE:

Let H be the set of all vectors of the form

$$\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$$

where a and b are arbitrary scalars. Show that H is not a vector space.

SOLUTION:

H is not a vector space, since $\bar{0} \notin H$ (the second entry is always nonzero).

EXAMPLE:

Let H be the set of all vectors of the form:

$$(a) \quad \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

$$(b) \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{where} \quad a \geq 0$$

$$(c) \quad \begin{bmatrix} a \\ a \\ 1 \end{bmatrix}$$

$$(d) \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{where} \quad a + 2b - c = 0$$

$$(e) \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{where} \quad 2a - b + c = 1.$$

Is H a subspace of R^3?

SOLUTION:

(a) We have

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which is a linear combination and therefore a subspace of R^3.

(b) It is not a subspace, since it is not closed under multiplication by a scalar. In fact,

$$(-1) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix},$$

and the first entry is negative, which contradicts the initial assumption.

(c) It is not a subspace, since there is no a zero vector (the last entry is always nonzero).

(d) We have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+2b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

which is a linear combination and therefore a subspace of R^3.

(e) We have

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 1 - 2a + b \end{bmatrix},$$

which is not a subspace, since there is no a zero vector. In fact,

$$\begin{bmatrix} a \\ b \\ 1 - 2a + b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if and only if $a = 0$ and $b = 0$. But in this case the last entry in nonzero, since

$$1 - 2a + b = 1 - 2 \cdot 0 + 0 = 1.$$

Contradiction.

DEFINITION:

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation

$$A\bar{x} = \bar{0}.$$

DEFINITION′:

The null space of an $m \times n$ matrix A is the set of all $\bar{x}$ in R^n that are mapped into the zero vector $\bar{0}$ in R^m by the linear transformation

$$\bar{x} \longmapsto A\bar{x}.$$

EXAMPLE:

Let

$$A=\begin{bmatrix}1 & -2 & -1\\ 2 & -3 & -4\end{bmatrix}.$$

Determine if $\bar{u}=\begin{bmatrix}5\\ 2\\ 1\end{bmatrix}$ belongs to the null space of A.

SOLUTION:

Since

$$A\bar{u}=\begin{bmatrix}1 & -2 & -1\\ 2 & -3 & -4\end{bmatrix}\begin{bmatrix}5\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix},$$

$\bar{u}$ is in Nul A.

EXAMPLE:

Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

Determine if $\bar{\boldsymbol{u}} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to the null space of $\boldsymbol{A}$.

SOLUTION:

Since

$$\boldsymbol{A}\bar{\boldsymbol{u}} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$\bar{\boldsymbol{u}}$ is in Nul $\boldsymbol{A}$.

THEOREM:

The null space of an $m \times n$ matrix A is a subspace of R^n. Equivalently, the set of all solutions to a system $A\bar{x} = \bar{0}$ of m homogeneous linear equations in n unknowns is a subspace of R^n.

EXAMPLE:

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

SOLUTION:

We find the general solution of $A\bar{x}=\bar{0}$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

therefore

$$\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0, \end{cases}$$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$

$$= x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\bar{u}} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\bar{w}},$$

so Nul A =Span $\{\bar{u}, \bar{v}, \bar{w}\}$.

DEFINITION:

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A.

REMARK:

So, if $A = [\bar{a}_1 \ldots \bar{a}_n]$, then

$$\text{Col } A = \text{Span}\{\bar{a}_1, \ldots, \bar{a}_n\}.$$

THEOREM:

The column space of an $m \times n$ matrix is a subspace of R^m.

EXAMPLE:

Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

Find a nonzero vector in Col A and a nonzero vector in Nul A.

SOLUTION:

1. Any column of A is a nonzero vector in Col A. For example, $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} =$

$$= 1 \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

2. To find a nonzero vector in Nul A, we row reduce the augmented matrix $[A\ \bar{0}]$:

$$[A\ \bar{0}] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

therefore any vector

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

is in Nul A. For example, if we put $x_3 = 1$, we get

$$\bar{u} = \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

is in Nul A.