DEFINITION:

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following 10 axioms (or rules):

1. **The sum of $\bar{u}$ and $\bar{v}$, denoted by $\bar{u} + \bar{v}$, is in V.**
2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.
3. $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$.
4. **There is a zero vector $\bar{0}$ in V such that $\bar{u} + \bar{0} = \bar{u}$.**
5. **For each $\bar{u}$ in V, there is a vector $-\bar{u}$ in V such that $\bar{u} + (-\bar{u}) = \bar{0}$.**
6. **The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.**
7. $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$.
8. $(c + d)\bar{u} = c\bar{u} + d\bar{u}$.
9. $c(d\bar{u}) = (cd)\bar{u}$.
10. $1 \cdot \bar{u} = \bar{u}$.

These axioms must hold for all vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ in V and all scalars c and d.

EXAMPLE:

R^n is a vector space. In fact, let

$$\bar{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad \bar{w} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Then

1. $\bar{u} + \bar{v}$ is in V.
2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.
3. $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$.
4. There is the zero vector $\bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

in V such that $\bar{u} + \bar{0} = \bar{u}$, since

$$\bar{u} + \bar{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{u}.$$

5. For each $\bar{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ **in** V**, there is the vector** $-\bar{u} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ **in** V **such that** $\bar{u} + (-\bar{u}) = \bar{0}$**, since**

$$\bar{u}+(-\bar{u}) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \bar{0}.$$

6. The scalar multiple of $\bar{u}$ **by** c**, denoted by** $c\bar{u}$**, is in** V**.**

7. $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$.

8. $(c + d)\bar{u} = c\bar{u} + d\bar{u}$.

9. $c(d\bar{u}) = (cd)\bar{u}$.

10. $1 \cdot \bar{u} = \bar{u}$.

EXAMPLE:

The set of all $n \times m$ matrices, i.e.

$$\begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots & & \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$$

Then

1. $\bar{u} + \bar{v}$ is in V.
2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.
3. $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$.
4. There is the zero vector $\bar{0} = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \dots & & \\ 0 & \dots & 0 \end{bmatrix}$

in V such that $\bar{u} + \bar{0} = \bar{u}$.

5. For each $\bar{u} = \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \dots \\ x_{n1} & \dots & x_{nm} \end{bmatrix}$ in V,

there is the vector $-\bar{u} = \begin{bmatrix} -x_{11} & \dots & -x_{1m} \\ -x_{21} & \dots & -x_{2m} \\ \dots \\ -x_{n1} & \dots & -x_{nm} \end{bmatrix}$ in V such that $\bar{u} + (-\bar{u}) = \bar{0}$.

6. The scalar multiple of $\bar{u}$ by c, denoted by $c\bar{u}$, is in V.

7. $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$.

8. $(c + d)\bar{u} = c\bar{u} + d\bar{u}$.

9. $c(d\bar{u}) = (cd)\bar{u}$.

10. $1 \cdot \bar{u} = \bar{u}$.

EXAMPLE:

The set P_n of polynomials of degree at most n:

$$\bar{p}(t) = a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, \ldots, a_0$ and the variable t are real numbers.

EXAMPLE:

The set of all real-valued functions defined on R.

DEFINITION:

A subspace of a vector space V is a subset H of V that has 3 properties:

1. The zero vector of V is in H.

2. H is closed under vector addition. That is, for each $\bar{u}$ and $\bar{v}$ in H, the sum $\bar{u} + \bar{v}$ is in H.

3. H is closed under multiplication by scalars. That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H.

EXAMPLE:

The set consisting of only the zero vector $\bar{0}$ in a vector space V is a subspace of V, called the zero subspace and written as $\{\bar{0}\}$.

WARNING:

R^2 is not a subspace of R^3, because R^2 is not a subset of R^3.

EXAMPLE:

The set

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real numbers} \right\}$$

is a subspace of R^3.

EXAMPLE:

Given $\bar{v}_1$ and $\bar{v}_2$ in a vector space V, let $H = \text{Span } \{\bar{v}_1, \bar{v}_2\}$. Show that H is a subspace of V.

SOLUTION:

First of all, note that

Span $\{\bar{v_1}, \bar{v_2}\} = \{\alpha\bar{v_1} + \beta\bar{v_2} : \alpha, \beta \in R\}$.

Therefore Span $\{\bar{v_1}, \bar{v_2}\}$ **is a subset of** V. **Moreover,**

1. **The zero vector** $\bar{0}$ **is in** H**, since**

$$\bar{0} = 0 \cdot \bar{v_1} + 0 \cdot \bar{v_2}.$$

2. H **is closed under vector addition. In fact, let**

$$\bar{u} = s_1\bar{v}_1 + s_2\bar{v}_2, \quad \bar{w} = t_1\bar{v}_1 + t_2\bar{v}_2.$$

By Axioms 2, 3, and 8 we have:

$$\bar{u} + \bar{w} = (s_1\bar{v}_1 + s_2\bar{v}_2) + (t_1\bar{v}_1 + t_2\bar{v}_2)$$
$$(s_1 + t_1)\bar{v}_1 + (s_2 + t_2)\bar{v}_2,$$

therefore $\bar{u} + \bar{w}$ **is in** H.

3. **Similarly, if** c **is any scalar, then by Axioms 7 and 9 we get**

$$c\bar{u} = c(s_1\bar{v}_1 + s_2\bar{v}_2) = (cs_1)\bar{v}_1 + (cs_2)\bar{v}_2,$$

therefore $c\bar{u}$ **is also in** H.
Thus, H **is a subspace of** V.

<u>THEOREM:</u>

If $\bar{v}_1, \ldots, \bar{v}_p$ are in a vector space V, then Span $\{\bar{v}_1, \ldots, \bar{v}_p\}$ is a subspace of V.