LINEAR TRANSFORMATIONS

DEFINITION:

A transformation (or function, or mapping) T from R^n to R^m is a rule that assigns to each vector $\bar{x}$ from R^n a vector $T(\bar{x})$ in R^m. The set R^n is called the domain, and R^m is called the codomain of T.

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$T : R^2 \to R^3, \quad T(\bar{x}) = A\bar{x}.$$

Then

$$T(\bar{u}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

DEFINITION:

A transformation T is linear if:

(i) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in the domain of T

(ii) $T(c\bar{u}) = cT(\bar{u})$ for all $\bar{u}$ in the domain of T and all scalars c

THEOREM:

A transformation T is linear if and only if

$$T(\bar{x}) = A\bar{x},$$

where A is a matrix. Moreover, for a given transformation $T : R^n \to R^m$ this matrix is unique.

THEOREM:

(a) Let $T : R^2 \to R^2$ be a linear transformation T determined by a 2×2 matrix A. If S is a region in R^2, then

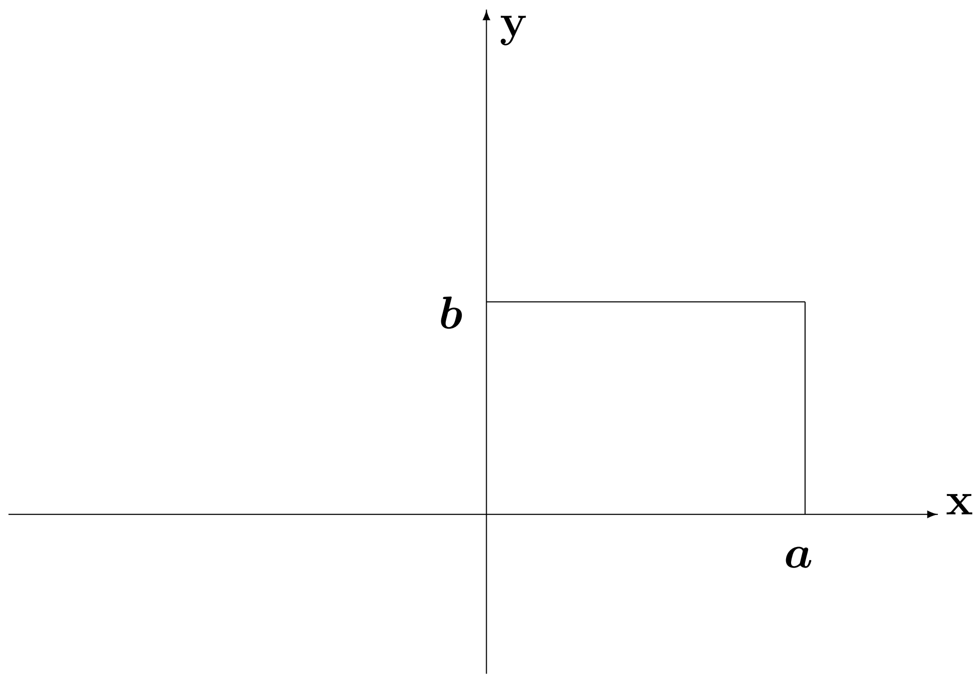
$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

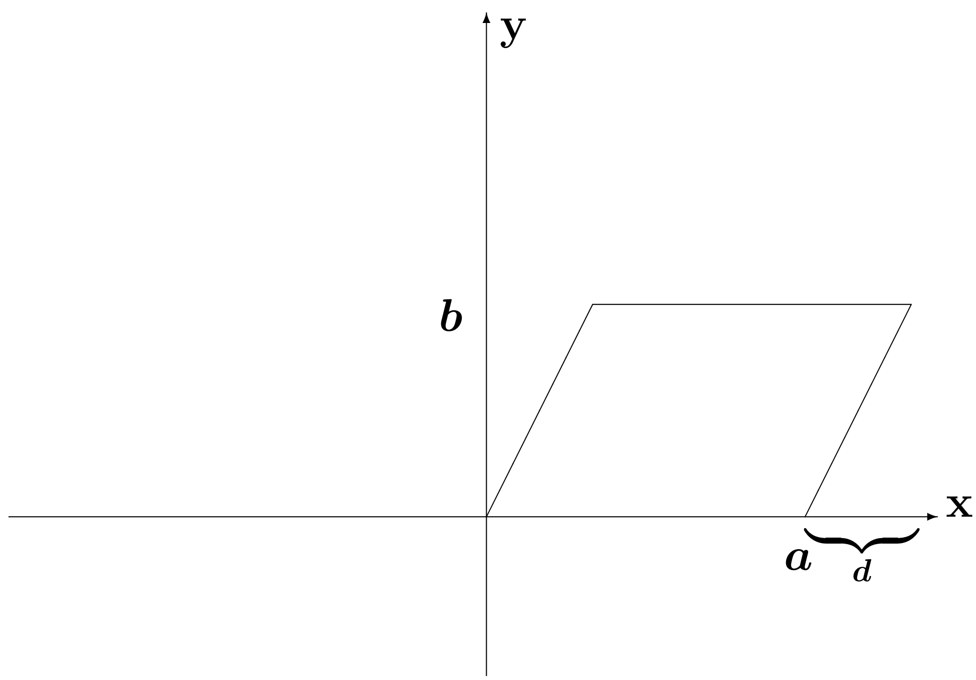
(b) Let $T : R^3 \to R^3$ be a linear transformation T determined by a 3×3 matrix A. If S is a region in R^3, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

EXAMPLE:

Find the area of the region:





PROBLEM:

Suppose an economy consists of the Coal, Electric Power, and Steel. Suppose that:

Coal sells 60% to Electric Power and 40% to Steel.

Electric Power sells 40% to Coal and 50% to Steel.

Steel sells 60% to Coal and 20% to Electric.

If possible, find equilibrium prices that make each sector's income match its expenditures.

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$P_C = .94 P_S$$

$$P_E = .85 P_S.$$

For example, if $P_S = \$100$ million, then $P_C = \$94$ million and $P_E = \$85$ million.

PROBLEM:

Suppose an economy consists of the Chemicals & Metals, Fuels & Power, and Machinery. Suppose that:

Chemicals sells 30% to Fuels and 50% to Machinery.

Fuels sells 80% to Chemicals and 10% to Machinery.

Machinery sells 40% to Chemicals and 40% to Fuels.

If possible, find equilibrium prices that make each sector's income match its expenditures.

$$\begin{bmatrix} .8 & -.8 & -.4 & 0 \\ -.3 & .9 & -.4 & 0 \\ -.5 & -.1 & .8 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & -.5 - \dfrac{5.5}{6} & 0 \\ 0 & 1 & -\dfrac{5.5}{6} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$P_C = \left(.5 + \frac{5.5}{6}\right) P_M$$

$$P_E = \frac{5.5}{6} P_M.$$

For example, if $P_M = \$100$ million, then $P_C \approx \$141.7$ m. and $P_E \approx \$91.7$ m.