PROBLEM:

Let $\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

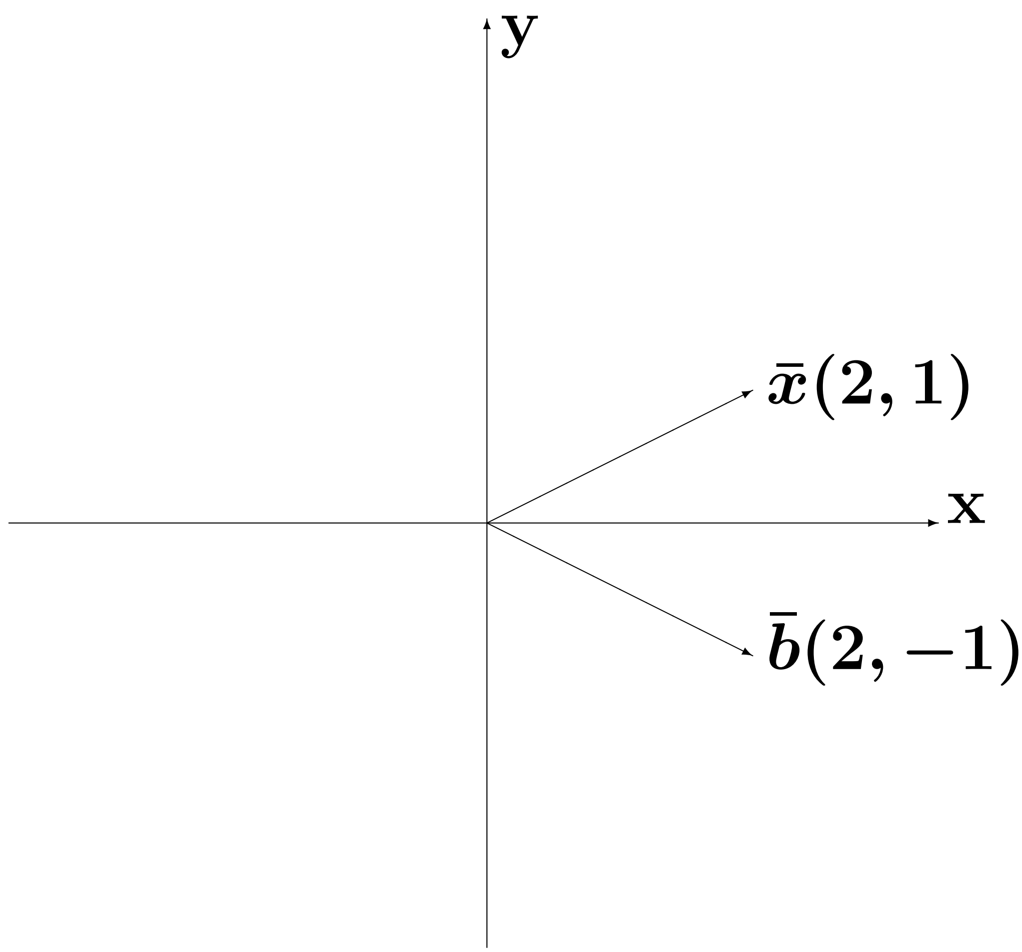
$$A_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

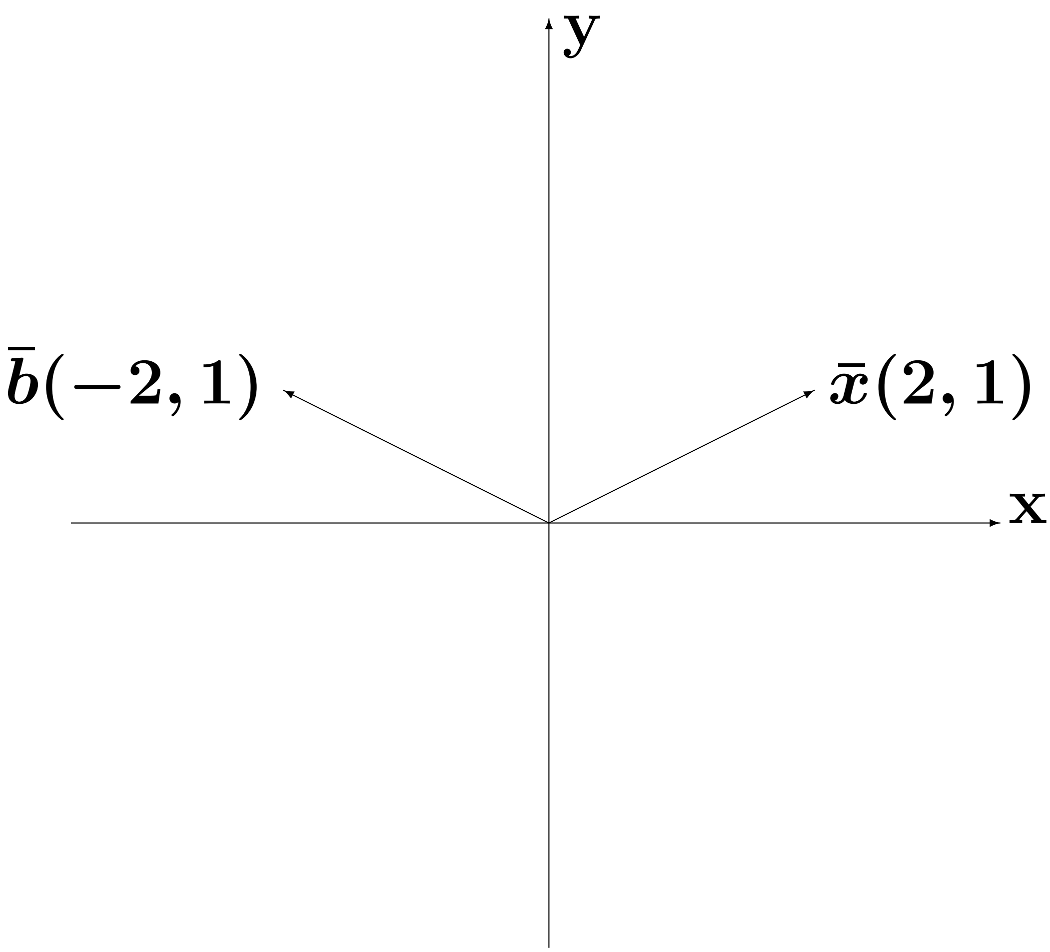
Find $A_i\bar{x}$, $B_i\bar{x}$, $C_i\bar{x}$. Provide illustrations and geometric explanations.

$$\mathbf{1.}\ \boldsymbol{A_1\bar{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$



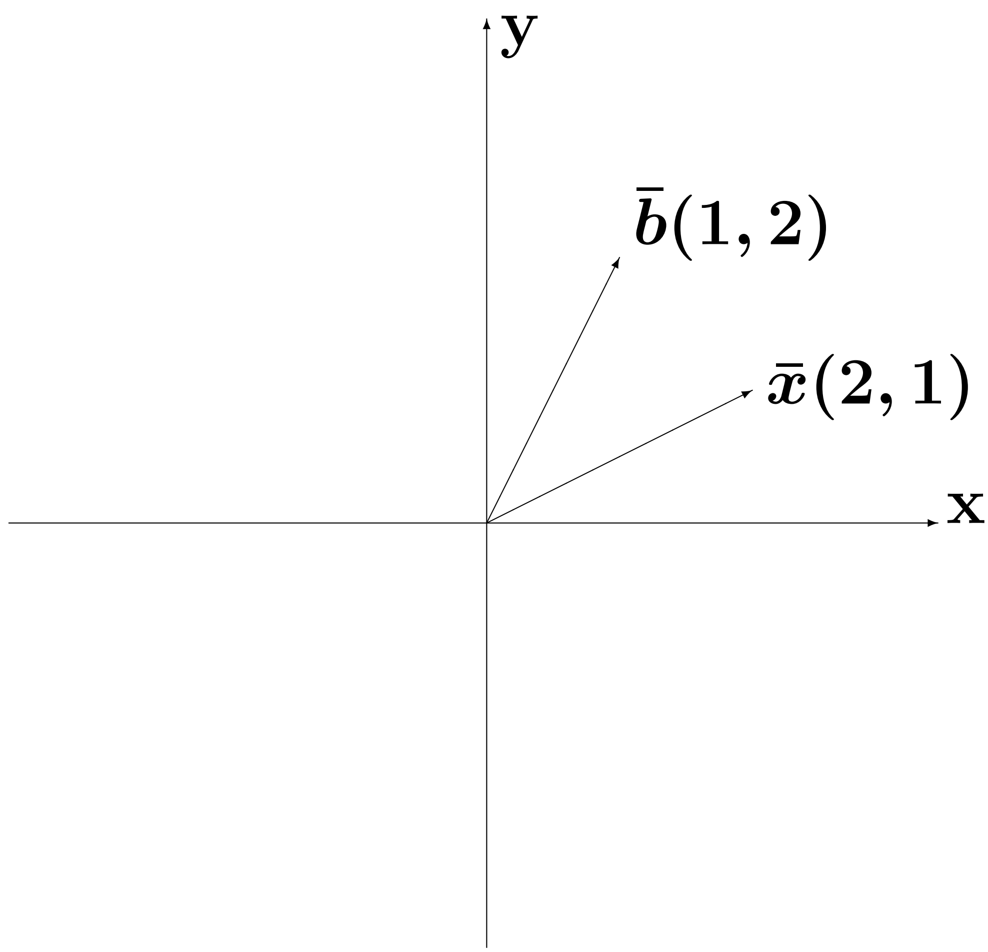
Reflection through the x-axis

$$2.\ \ A_2\bar{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



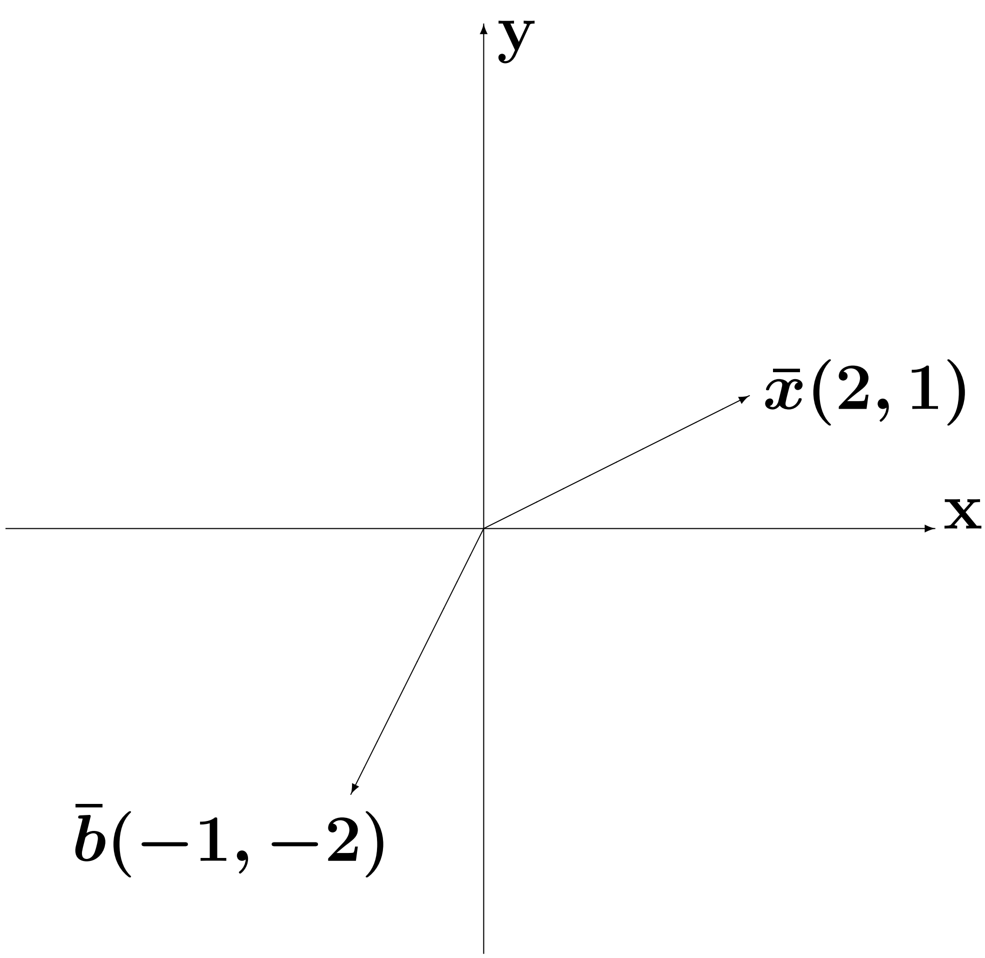
Reflection through the y-axis

$$\mathbf{3.}\ \ \boldsymbol{A_3\bar{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}$$



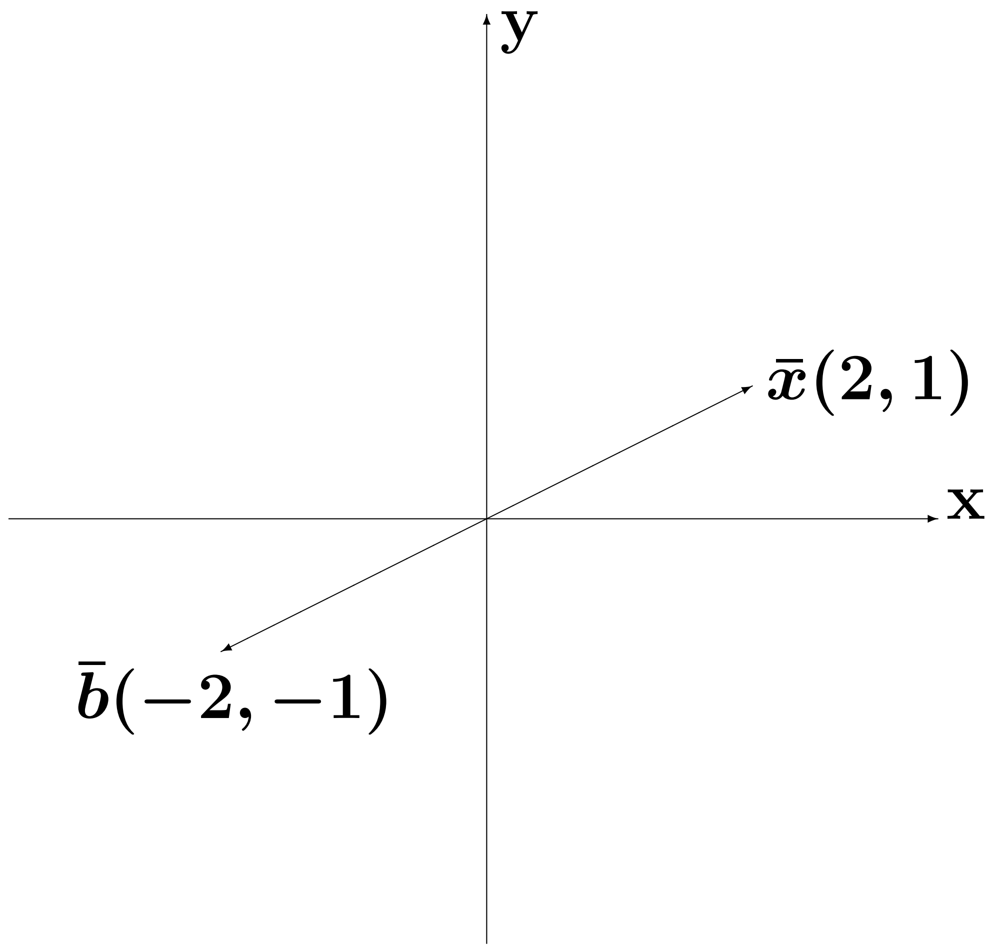
Reflection through the line $y = x$

$$\mathbf{4.}\ A_4\bar{x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$



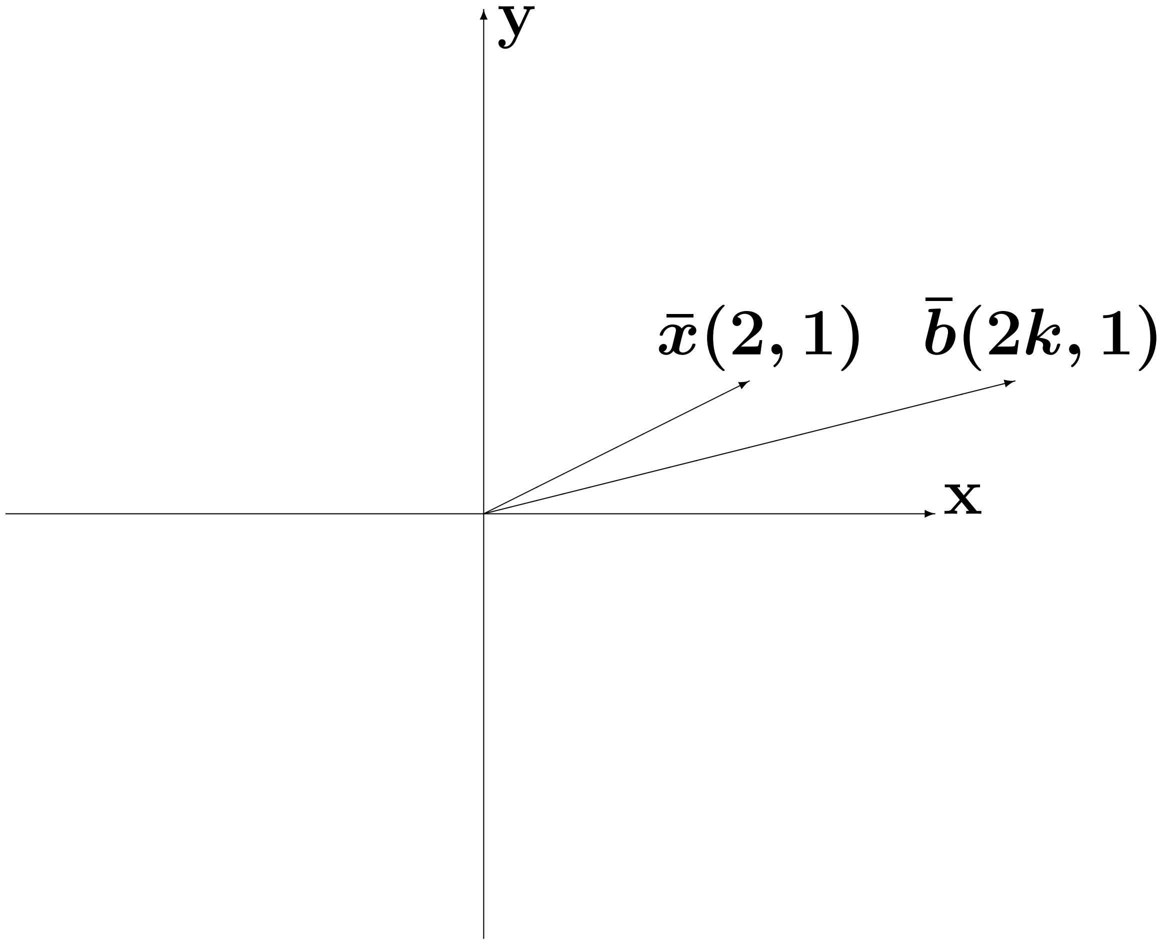
Reflection through the line $y = -x$

$$\mathbf{5.}\quad A_5\bar{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



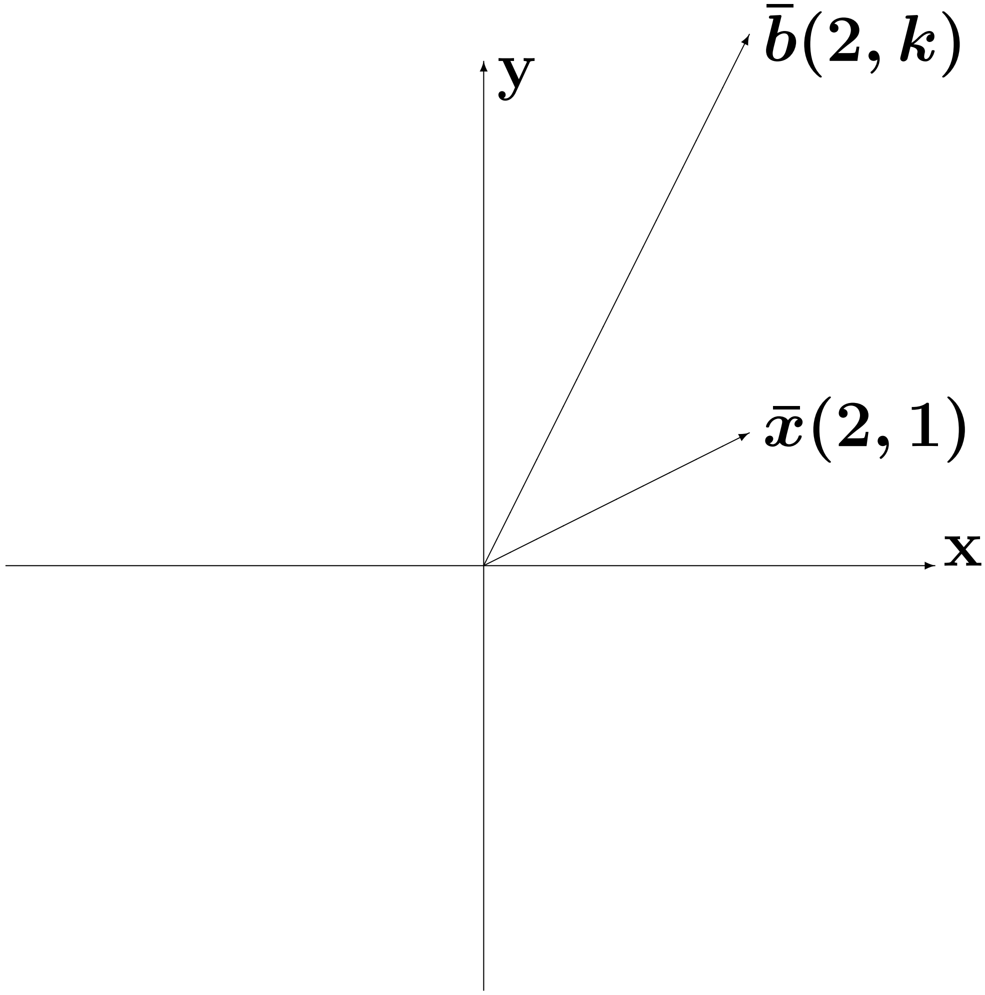
Reflection through the origin

$$\textbf{6. } B_1\bar{x} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2k \\ 1 \end{bmatrix}$$



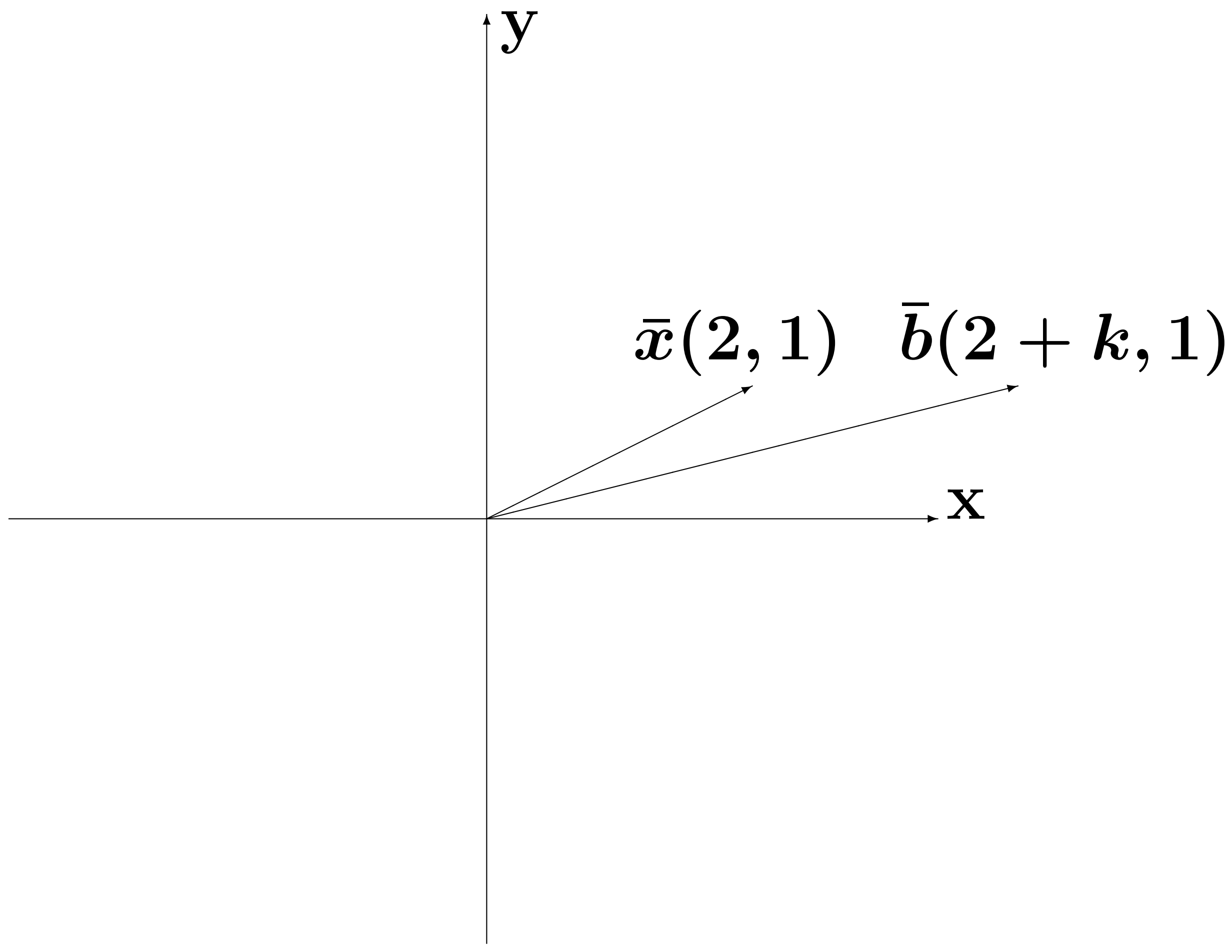
Horizontal expansion

7. $$B_2\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ k \end{bmatrix}$$



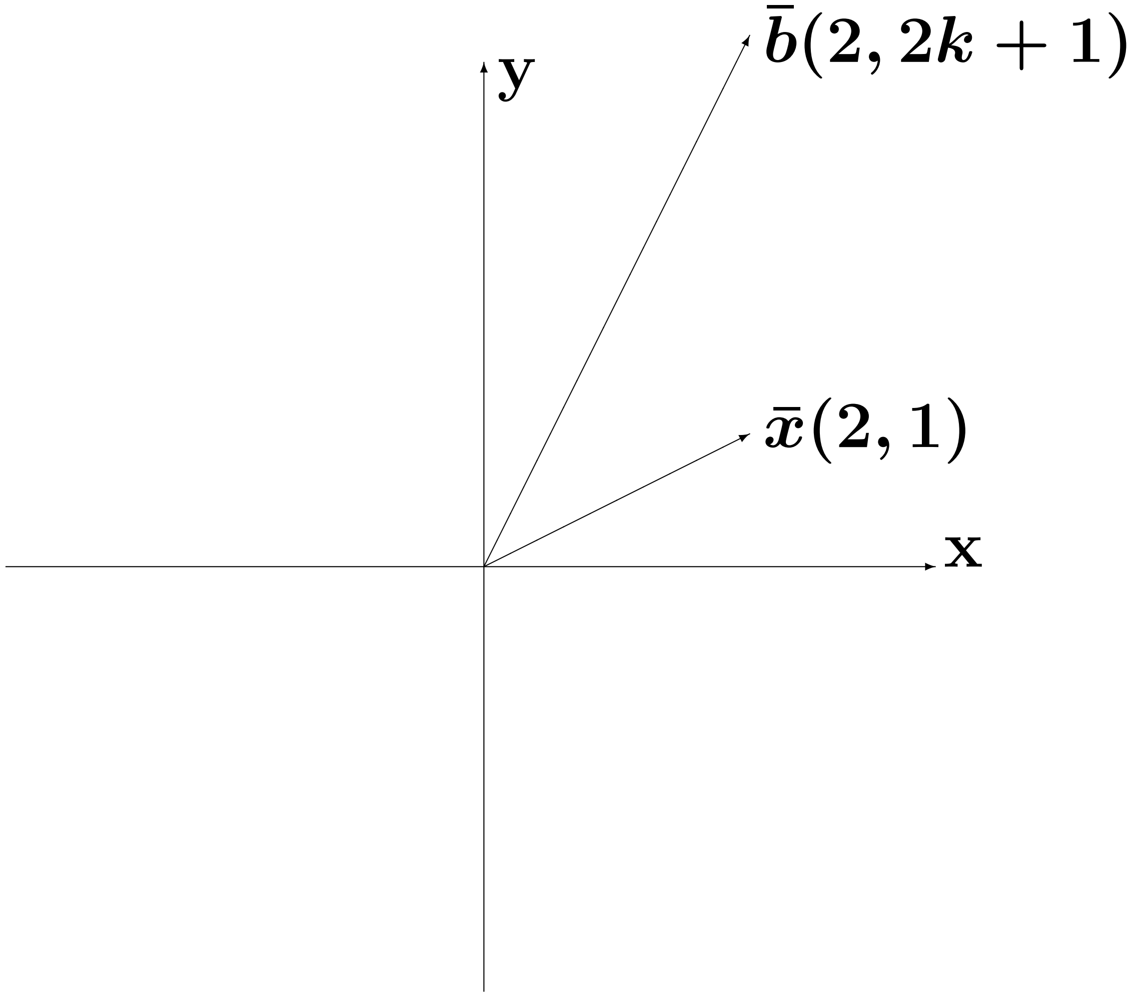
Vertical expansion

$$\textbf{8. } C_1\bar{x} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+k \\ 1 \end{bmatrix}$$



Horizontal shear

$$9.\ \ C_2\bar{x} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2k+1 \end{bmatrix}$$



Vertical shear

LINEAR TRANSFORMATIONS

DEFINITION:

A transformation (or function, or mapping) T from R^n to R^m is a rule that assigns to each vector $\bar{x}$ from R^n a vector $T(\bar{x})$ in R^m. The set R^n is called the domain, and R^m is called the codomain of T.

EXAMPLE:

Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

$$T : R^2 \rightarrow R^3, \quad T(\bar{x}) = A\bar{x}.$$

Find $T(\bar{u})$.

<u>SOLUTION:</u>

We have:

$$\boldsymbol{T(\bar{u})} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} \\ \mathbf{3} & \mathbf{5} \\ \mathbf{-1} & \mathbf{7} \end{bmatrix} \begin{bmatrix} \mathbf{2} \\ \mathbf{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{5} \\ \mathbf{1} \\ \mathbf{-9} \end{bmatrix}.$$

DEFINITION:

A transformation T is linear if:

(*i*) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}, \bar{v}$ in the domain of T

(*ii*) $T(c\bar{u}) = cT(\bar{u})$ for all $\bar{u}$ in the domain of T and all scalars c

THEOREM:

A transformation T is linear if and only if

$$T(\bar{x}) = A\bar{x},$$

where A is a matrix. Moreover, for a given transformation $T : R^n \rightarrow R^m$ this matrix is unique.

PROBLEM:

Find a matrix A such that for any $\bar{x}$ from R^2 we have

$$T(\bar{x}) = 3\bar{x}.$$

SOLUTION:

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$T(\bar{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

and

$$3\bar{x} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}.$$

Hence,

$$a = 3,\ b = 0,\ c = 0,\ d = 3,$$

so

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$