

PROBLEM: Solve

$$\begin{cases} 3x_1 - 5x_2 = 7 \\ 8x_1 + 9x_2 = -5 \end{cases}$$

SOLUTION: We have

$$x_1 = \frac{\begin{vmatrix} 7 & -5 \\ -5 & 9 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 8 & 9 \end{vmatrix}} = \frac{63 - 25}{27 + 40} = \frac{38}{67}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 7 \\ 8 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 8 & 9 \end{vmatrix}} = \frac{-15 - 56}{67} = -\frac{71}{67}$$

PROBLEM: Solve

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ x_1 - x_2 - x_3 = 3 \end{cases}$$

SOLUTION: We have

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}} = ???$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

DEFINITION:

The determinant of an $n \times n$ matrix A is the following sum:

$$\begin{aligned} \det A = & a_{11} \det A_{11} - a_{12} \det A_{12} \\ & + a_{13} \det A_{13} \\ & - \dots \\ & + (-1)^{n+1} a_{1n} \det A_{1n}, \end{aligned}$$

where A_{1j} are submatrices formed by deleting from A the first row and j th column.

PROBLEM: Find

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

THEOREM:

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A .

EXAMPLE:

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 24$$

THEOREM:

We have $\det A = 0$

(a) if A contains a zero-row or zero-column.

EXAMPLE:
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(b) if A contains two similar rows or columns.

EXAMPLE:
$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$$

(c) if some row (column) of A is a multiple of some other row (column) of A .

EXAMPLE:
$$\begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

THEOREM:

Let A be a square matrix.

(a) If a multiple of one row (column) of A is added to another row (column) to produce a matrix B , then $\det A = \det B$.

EXAMPLE:
$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2.$$

(b) If two rows (columns) of A are interchanged to produce B , then $\det A = -\det B$.

EXAMPLE:
$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 3 & 3 & 8 \end{vmatrix}$$

(c) If one row (column) of A is multiplied by k to produce B , then $\det B = k \det A$.

EXAMPLE:
$$\begin{vmatrix} 100 & 300 \\ 1 & 2 \end{vmatrix} = 100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}.$$

PROBLEM: Find

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix}$$

THEOREM:

Let A be a square matrix. Then

(a) $\det A^T = \det A.$

(b) $\det(AB) = \det A \det B.$