ADDITION:

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -4 & -1 & -3 \end{bmatrix}$$

MULTIPLICATION BY A SCALAR:

$$(-2) \begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 6 \\ 2 & 0 & 4 \end{bmatrix}$$

PRODUCT:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$$

TRANSPOSITION:

$$A = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix} \qquad A^T = \begin{bmatrix} -3 & 4 & 8 \\ 1 & 7 & -5 \end{bmatrix}$$

WARNING:

In general, $AB \neq BA$.

EXAMPLE:

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

INVERSE:

Let A be an $n \times n$ matrix. Then a matrix A^{-1} with

$$A^{-1}A = AA^{-1} = I$$

is said to be an inverse of A.

EXAMPLE:

If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, since

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

<u>THEOREM</u>:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

<u>EXAMPLE</u>:

If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, since

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

PROBLEM:

Solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + 4x_2 = 6 \end{cases}$$

SOLUTION:

We have:

$$A\bar{x} = B \Rightarrow \bar{x} = A^{-1}B,$$

therefore

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

<u>PROPERTIES</u>:

Let A and B be invertible $n \times n$ matrices. Then

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(A^T)^{-1} = (A^{-1})^T$

THEOREM:

Let A be a square $n \times n$ matrix. Then the following statements are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\bar{x} = \bar{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n.

(g) The columns of A span R^n.

(h) A^T is an invertible matrix.

ALGORITHM FOR FINDING A^{-1}:

1. Row reduce the augmented matrix $[A\ I]$.

2. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$.

3. Otherwise, A does not have an inverse

EXAMPLE:

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$. Find A^{-1}.

PROBLEM:

Let $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$. **Find** $\boldsymbol{A}^{-1}$.