DEFINITION:

If A and B are $m \times n$ matrices, then the sum $A+B$ is the $m\times n$ matrix whose entries are the sums of the corresponding entries of A and B.

EXAMPLE:

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -4 & -1 & -3 \end{bmatrix}$$

REMARK: We can add matrices only of the same size.

EXAMPLE:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} = ???$$

DEFINITION:

If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

EXAMPLE:

$$(-2)\begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 6 \\ 2 & 0 & 4 \end{bmatrix}$$

PROPERTIES:

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $r(A + B) = rA + rB$

(d) $(r + s)A = rA + sA$

(e) $r(sA) = (rs)A$

PROBLEM:

Consider the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}.$$

If possible, compute:

(a) $\boldsymbol{AB}$

(b) $\boldsymbol{AC} + \boldsymbol{B}^2$

(c) $\boldsymbol{AB} + \boldsymbol{C}^2$

SOLUTION:

We have:

(a) $\boldsymbol{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}.$

(b) **Impossible.**

(c) $\boldsymbol{AB + C^2}$

$$= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -16 \\ 16 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & -8 \\ 32 & 30 \end{bmatrix}.$$

PROPERTIES:

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(B + C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$

WARNING

1. In general, $AB \neq BA$.

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. **Then**

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So,

$$AB \neq BA.$$

WARNING

2. If $AB = AC$, then it is not true in general that $B = C$.

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = AC, \quad \text{but} \quad B \neq C.$$

WARNING

3. If $AB = 0$, then it is not true in general that $A = 0$ or $B = 0$.

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. **Then**

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = 0, \quad \text{but} \quad A \neq 0 \quad \text{and} \quad B \neq 0.$$

THE TRANSPOSE OF A MATRIX

DEFINITION:

Let A be an $m \times n$ matrix. The transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix} \qquad B^T = \begin{bmatrix} -3 & 4 & 8 \\ 1 & 7 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

PROPERTIES:

Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$ for any scalar r

(d) $(AB)^T = B^T A^T$

THE INVERSE OF A MATRIX

DEFINITION:

The identity matrix I is the $n \times n$ matrix of the form

$$I = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

MAIN PROPERTY:

$$AI = IA = A$$

DEFINITION:

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I.$$

In this case, C is an inverse of A and is denoted by A^{-1}. So,

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. **Then** $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.
In fact, we have

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

THEOREM:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$**. If** $ad - bc \neq 0$**, then** A **is invertible and**

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$**, then** A **is not invertible.**

PROBLEM:

Solve the following system of equations:

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + 4x_2 = 6 \end{cases}$$

SOLUTION:

We have:

$$\boldsymbol{A\bar{x} = B \Rightarrow \bar{x} = A^{-1}B,}$$

therefore

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$