DEFINITION:

A system of linear equations is said to be homogeneous if it can be written in the form $A\bar{x} = \bar{0}$. Otherwise, it is non-homogeneous.

EXAMPLE:

$$\begin{cases} 3x_1 + 5x_2 = 0 \\ 6x_1 + 2x_2 = 0 \end{cases}$$ **HOMOGENEOUS**

$$\begin{cases} 3x_1 + 5x_2 = 1 \\ 6x_1 + 2x_2 = 0 \end{cases}$$ **NONHOMOGEN.**

<u>THEOREM</u>:

Suppose the equation $A\bar{x} = \bar{b}$ is consistent for some given $\bar{b}$, and let $\bar{p}$ be a solution. Then the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form

$$\bar{w} = \bar{p} + \bar{v}_h,$$

where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$.

THEOREM:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution.

2. There are no free variables.

3. Number of columns of A = Number of pivot positions.

DEFINITION:
Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are said to be linearly dependent if there exist scalars $c_1, \ldots, c_p$, not all zero, such that

$$c_1\bar{v}_1 + \ldots + c_p\bar{v}_p = \bar{0}.$$

Vectors $\bar{v}_1, \ldots, \bar{v}_p$ are said to be linearly independent if the vector equation

$$c_1\bar{v}_1 + \ldots + c_p\bar{v}_p = \bar{0}$$

has only the trivial solution.

EXAMPLE: Vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

are linearly dependent.

Vectors

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

are linearly independent.

<u>THEOREM</u>:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. **A homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution.**
2. **There are no free variables.**
3. **Number of columns of A = Number of pivot positions.**
4. **The columns of a matrix A are linearly independent.**

THEOREM:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. A homogeneous system $A\bar{x} = \bar{0}$ has a nontrivial solution.
2. There are free variables.
3. Number of columns of $A >$ Number of pivot positions.
4. The columns of a matrix A are linearly dependent.
5. At least one column of A is a linear combination of other columns.

1. Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

2. Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

3. Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ span R^3 ?

4. Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span R^3 ?

1. Let

$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

2. Let

$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

3. Let

$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \ \bar{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}, \ \bar{v}_4 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ span R^3 ?

4. Let

$$\bar{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \ \bar{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}.$$

(a) Are $\bar{v}_1, \bar{v}_2$ linearly independent?
(b) Does $\{\bar{v}_1, \bar{v}_2\}$ span R^3 ?