$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 + 3x_2 = -5 \\ x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_3 = 3 \\ x_2 \text{ is free} \end{cases}$$

$$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

Let

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

(a) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^2 ?

(b) Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span R^3 ?

(c) Is $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$?

(d) Is $\begin{bmatrix} 7 \\ 32 \end{bmatrix}$ in Span $(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$?

(e) Does

$$x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

have a solution?

(f) Does

$$x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = \begin{bmatrix} 7 \\ 32 \end{bmatrix}$$

have a solution?

<u>THEOREM</u>:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. For each $\bar{b}$ in R^m, the equation $A\bar{x} = \bar{b}$ has a solution.

2. Each $\bar{b}$ in R^m is a linear combination of the columns of A.

3. The columns of A span R^m.

4. A has a pivot position in every row, i.e. A has m pivot positions.

Row-Vector Rule for Computing $A\bar{x}$:

If a product $A\bar{x}$ is defined, then the ith entry in $A\bar{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\bar{x}$.

EXAMPLE:

1. $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

2. $$\begin{bmatrix} 2 & -3 & -9 & 11 \\ -3 & 0 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - 3x_2 - 9x_3 + 11x_4 \\ -3x_1 + 6x_3 - 4x_4 \end{bmatrix}$$

3. $$\begin{bmatrix} 2 & -3 & -9 & 11 \\ -3 & 0 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ???$$

PROBLEM:

1. Write the general solution of the system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

in a vector form.

2. Solve the following system

$$\begin{cases} x_1 + 3x_2 + 4x_3 = 0 \\ 3x_1 + 9x_2 + 7x_3 = 0 \end{cases}$$

and write the general solution in a vector form.

<u>THEOREM</u>:

Suppose the equation $A\bar{x} = \bar{b}$ is consistent for some given $\bar{b}$, and let $\bar{p}$ be a solution. Then the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form

$$\bar{w} = \bar{p} + \bar{v}_h,$$

where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$.

PROBLEM:

Let

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \bar{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

1. Is $\bar{b}$ in the subset of R^3 spanned by the columns of A?

2. Do columns of A span R^3?

3. Describe all solutions of $A\bar{x} = \bar{b}$.