ECHELON FORM:

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

REDUCED ECHELON FORM:

$$\begin{bmatrix} 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

DEFINITION:
A pivot position in a matrix is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
A pivot column is a column of A that contains a pivot position.

ELEMENTARY ROW OPERATIONS:

1. Replace one row by the sum of itself and a multiple of another row.

2. Interchange two rows.

3. Multiply all entries in a row by a nonzero constant.

DEFINITION:
A pivot position in a matrix is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
A pivot column is a column of A that contains a pivot position.

DEFINITION:
The variables corresponding to pivot columns are called basic variables.
The other variables are called free variables.

ELEMENTARY ROW OPERATIONS:

1. Replace one row by the sum of itself and a multiple of another row.

2. Interchange two rows.

3. Multiply all entries in a row by a nonzero constant.

PROBLEMS:

Solve the following systems and indicate echelon forms, reduced echelon forms, pivot positions and pivot columns:

1.
$$\begin{cases} 2x_1 - x_2 = -1 \\ x_1 + 2x_2 - x_3 = -2 \\ x_2 + x_3 = -2 \end{cases}$$

2.
$$\begin{cases} 2x_1 + 3x_2 + 8x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \\ x_3 + 2x_4 = 1 \\ x_1 + x_4 = -24 \end{cases}$$

3*.
$$\begin{cases} 2x_1 + 5x_2 - 8x_3 = 8 \\ 4x_1 + 3x_2 - 9x_3 = 9 \\ 2x_1 + 3x_2 - 5x_3 = 7 \\ x_1 + 8x_2 - 7x_3 = 12 \end{cases}$$

4*.
$$\begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 - 2x_5 = -4 \\ x_1 + 6x_2 + 4x_3 - 13x_4 - 3x_5 = -1 \\ x_1 + 6x_2 + 2x_3 - 5x_4 - x_5 = 3 \end{cases}$$

$$\begin{cases} 2x_1 - x_2 = -1 \\ x_1 + 2x_2 - x_3 = -2 \\ x_2 + x_3 = -2 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & -1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 2 & 3 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & -5 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = -1 \\ x_2 = -1 \\ x_3 = -1 \end{cases}$$

PROBLEMS:

Find the general solutions of the following systems. Indicate augmented matrices, echelon forms, reduced echelon forms, pivot positions, pivot columns, basic variables, free variables:

1. $$\begin{cases} x_1 + 3x_2 + 4x_3 = 7 \\ 3x_1 + 9x_2 + 7x_3 = 6 \end{cases}$$

2. $$\begin{cases} 3x_1 - 4x_2 + 2x_3 = 0 \\ -9x_1 + 12x_2 - 6x_3 = 0 \\ -6x_1 + 8x_2 - 4x_3 = 0 \end{cases}$$