

SYSTEMS OF LINEAR EQUATIONS

$$\begin{cases} x_1 - 2x_2 = -1 \\ x_1 - x_2 = 3 \end{cases}$$

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$$\begin{cases} x_1 - 2x_2 = -1 \\ x_2 = 4 \end{cases}$$

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$$\begin{cases} x_1 = 7 \\ x_2 = 4 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases}$$

⇓

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ x_2 + 2x_3 = 8 \end{cases}$$

⇓

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ 4x_3 = 12 \end{cases}$$

⇓

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -1 & 2 \\ 1 & 2 & 3 & 14 \end{bmatrix}$$

$$\Downarrow$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ x_2 + 2x_3 = 8 \end{cases} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

$$\Downarrow$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ 4x_3 = 12 \end{cases} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 12 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -1 & 2 \\ 1 & 2 & 3 & 14 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ 4x_3 = 12 \end{cases} \Leftarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 12 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

ELEMENTARY ROW OPERATIONS:

- 1. Replace one row by the sum of itself and a multiple of another row.**
- 2. Interchange two rows.**
- 3. Multiply all entries in a row by a nonzero constant.**

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \Rightarrow \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = 5/2 \end{cases} \Leftarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

(Inconsistent System)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 4x_1 - 5x_2 - 9x_3 = 9 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 4 & -5 & -9 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -x_3 = -3 \end{cases} \Leftarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

(Consistent System)

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 4x_1 - 5x_2 - 9x_3 = 9 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 4 & -5 & -9 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\Leftarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

PROBLEMS:

Solve the following systems and indicate echelon forms, reduced echelon forms, pivot positions and pivot columns:

$$1. \begin{cases} 2x_1 - x_2 & = -1 \\ x_1 + 2x_2 - x_3 & = -2 \\ x_2 + x_3 & = -2 \end{cases}$$

$$2. \begin{cases} 2x_1 + 3x_2 + 8x_4 & = 0 \\ x_2 - x_3 + 3x_4 & = 0 \\ x_3 + 2x_4 & = 1 \\ x_1 + x_4 & = -24 \end{cases}$$

$$3^*. \begin{cases} 2x_1 + 5x_2 - 8x_3 & = 8 \\ 4x_1 + 3x_2 - 9x_3 & = 9 \\ 2x_1 + 3x_2 - 5x_3 & = 7 \\ x_1 + 8x_2 - 7x_3 & = 12 \end{cases}$$

$$4^*. \begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 - 2x_5 & = -4 \\ x_1 + 6x_2 + 4x_3 - 13x_4 - 3x_5 & = -1 \\ x_1 + 6x_2 + 2x_3 - 5x_4 - x_5 & = 3 \end{cases}$$

$$\begin{cases} 2x_1 - x_2 = -1 \\ x_1 + 2x_2 - x_3 = -2 \\ x_2 + x_3 = -2 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & -1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 2 & 3 \\ 0 & 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & -5 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = -1 \\ x_2 = -1 \\ x_3 = -1 \end{cases}$$

$$\begin{cases} 2x_1 + 3x_2 + 8x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \\ x_3 + 2x_4 = 1 \\ x_1 + x_4 = -24 \end{cases}$$

$$\Downarrow$$

$$\begin{array}{l} \sim \begin{bmatrix} 2 & 3 & 0 & 8 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & -24 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 3 & 0 & 6 & 48 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & -1 & 1 & -16 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & 16 \\ 0 & 0 & 0 & 3 & -15 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 26 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \end{array} \quad \begin{array}{l} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 2 & 3 & 0 & 8 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & 16 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -24 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -1 & 16 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \\ \Rightarrow \begin{cases} x_1 = -19 \\ x_2 = 26 \\ x_3 = 11 \\ x_4 = -5 \end{cases} \end{array}$$

$$\begin{cases} 2x_1 + 5x_2 - 8x_3 = 8 \\ 4x_1 + 3x_2 - 9x_3 = 9 \\ 2x_1 + 3x_2 - 5x_3 = 7 \\ x_1 + 8x_2 - 7x_3 = 12 \end{cases}$$

$$\Downarrow$$

$$\begin{aligned} & \begin{bmatrix} 2 & 5 & -8 & 8 \\ 4 & 3 & -9 & 9 \\ 2 & 3 & -5 & 7 \\ 1 & 8 & -7 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 4 & 3 & -9 & 9 \\ 2 & 3 & -5 & 7 \\ 2 & 5 & -8 & 8 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & -7 & 7 & -7 \\ 0 & -2 & 3 & -1 \\ 2 & 5 & -8 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & -7 & 7 & -7 \\ 0 & -2 & 3 & -1 \\ 0 & -11 & 6 & -16 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & 11 & -6 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 5 & 5 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 8 & -7 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 8 & 0 & 19 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 2 \\ x_3 = 1 \end{cases} \end{aligned}$$