SYSTEMS OF LINEAR EQUATIONS

DEFINITION 1:
A linear equation in the variables $x_1, \ldots,$ x_n is an equation that can be written in the form

$$a_1 x_1 + \ldots + a_n x_n = b,$$

where $a_1, \ldots, a_n$ and b are constants, $x_1, \ldots, x_n$ are variables.

EXAMPLE:
The equation

$$2x_1 + x_2 - 7x_3 = \sqrt{5}$$

is linear.
The equation

$$3x_1 x_2 + 2x_3^2 = 1$$

is NOT linear.

DEFINITION 2:
A system of linear equations (or a linear system) is a collection of one or more linear equations.

EXAMPLE:

$$\begin{cases} 3x_1 + 2x_2 + 7x_3 - x_4 = 6 \\ x_1 + x_2 - x_3 + x_4 = 1 \\ 4x_1 + 3x_2 + 6x_3 = 8 \end{cases}$$

SOLVING A LINEAR SYSTEM

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 - 2x_2 = -1 \\ x_2 = 2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ x_2 + 2x_3 = 8 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ 4x_3 = 12 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ x_3 = 3 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -1 & 2 \\ 1 & 2 & 3 & 14 \end{bmatrix}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ x_2 + 2x_3 = 8 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ 4x_3 = 12 \end{cases} \qquad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 12 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 + 2x_2 + 3x_3 = 14 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -1 & 2 \\ 1 & 2 & 3 & 14 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 2 & 8 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_2 - 2x_3 = -4 \\ 4x_3 = 12 \end{cases} \Leftarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 12 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 4x_1 - 5x_2 - 9x_3 = 9 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 4 & -5 & -9 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 3 & -13 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ - x_3 = -3 \end{cases} \Leftarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases}$$

$$\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases} \Rightarrow \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = 5/2 \end{cases} \Leftarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

CONTRADICTION !

DEFINITION 3:
A matrix is a rectangular array.

EXAMPLE:
Given a system

$$\begin{cases} \phantom{2x_1 - {}} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 5x_1 - 8x_2 + 7x_3 = 1 \end{cases}$$

the matrix

$$\begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix}$$

is called the coefficient matrix of the system, and

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

is called the augmented matrix of the system.

ELEMENTARY ROW OPERATIONS:

1. Replace one row by the sum of itself and a multiple of another row.

2. Interchange two rows.

3. Multiply all entries in a row by a nonzero constant.

PROBLEMS:
Solve the following systems:

1. $$\begin{cases} x_1 \qquad\qquad - 3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ \qquad\quad x_2 + 5x_3 = -2 \end{cases}$$

2. $$\begin{cases} \qquad\quad x_2 + 4x_3 = -5 \\ x_1 + 3x_2 + 5x_3 = -2 \\ 3x_1 + 7x_2 + 7x_3 = 6 \end{cases}$$