M340L - Matrices and Matrix Calculations - Fall 2002

Final Exam, December 11, 2002

I. (30 points) Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 2 \\ -2 & 3 & -1 & 0 & 2 & -3 \\ 3 & 4 & 1 & 8 & 5 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find all solutions to the system of linear equations $A\mathbf{x} = \mathbf{b}$, write the solution set in the parametric form and indicate basic and free variables, particular solution of the system, general solution of a correspondent homogeneous system;

(b) Find bases for Col A and Nul A;

(c) Find the rank of A;

PROVIDE CLEAR EXPLANATIONS.

II. (20 points) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Find the least squares solution of the system of linear equations $A\mathbf{x} = \mathbf{b}$. Show all steps and tell what you do.

III. (30 points)

(a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 9 & -13 \\ 5 & -14 & 20 \end{bmatrix}$. Explain your work.

(b) Evaluate the determinant $D = \begin{vmatrix} 5 & 2 & 4 & 1 & 1 \\ 16 & 6 & 12 & 3 & 3 \\ 50 & -2 & -5 & -1 & -1 \\ 60 & 4 & 101 & 2 & 5 \\ 70 & -3 & 102 & -1 & 2005 \end{vmatrix}$. Explain your work.

IV. (30 points) Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$.

(a) Find all eigenvectors of A;

(b) Diagonalize A.

PROVIDE CLEAR EXPLANATIONS.

V. (30 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_5$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $\mathbf{p}(t) - 2t\mathbf{p}(t) - 3t^3\mathbf{p}(t)$.

(a) Find the image of $\mathbf{p}(t) = t^2 - t - 8$;

(b) Show that T is linear;

(c) Find the matrix for T relative to the standard bases;

(d) Show that $\{1 + t^2,\ 3,\ 5 - t\}$ is the basis for $\mathbb{P}_2$;

(e) Find the matrix for T relative to the bases $\{1 + t^2,\ 3,\ 5 - t\}$ and $\{1,\ t,\ t^2,\ t^3,\ t^4,\ t^5\}$.

PROVIDE CLEAR EXPLANATIONS.

VI. (30 points) Let $V = \mathbb{R}^3$. Determine if the given set is a subspace of V. Justify your answer.

(a) The set of vectors of the form $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$, where a, b are in $\mathbb{R}$.

(b) The set of vectors of the form $\begin{bmatrix} a \\ b \\ a+2b \end{bmatrix}$, where a, b are in $\mathbb{R}$ and $a > 0$.

(c) The set of vectors of the form $\begin{bmatrix} a \\ a \\ c \end{bmatrix}$, where a, c are in $\mathbb{R}$.

(d) The set of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where a, b, c are in $\mathbb{R}$ and $2a - b + c = 1$.

VII. (30 points) Let $V = \mathbb{P}_2$. Let also

$$\mathbf{p_1}(t) = 1 + 4t - 2t^2, \quad \mathbf{p_2}(t) = -2 - 3t + 7t^2, \quad \mathbf{p_3}(t) = 4 + t + ht^2$$

(a) For what value(s) of h is $\mathbf{p_3}(t)$ in Span$\{\mathbf{p_1}(t),\ \mathbf{p_2}(t)\}$. Explain your work and justify your conclusions.

(b) For what value(s) of h does not vectors $\mathbf{p_1}(t),\ \mathbf{p_2}(t),\ \mathbf{p_3}(t)$ span V. Explain your work and justify your conclusions.

(c) Construct an orthogonal basis for Span$\{\mathbf{p_1}(t),\ \mathbf{p_2}(t)\}$ using the Gram-Schmidt process. Explain your work and justify your conclusions.