Math175 - Discrete Mathematics - Spring 2005

Quiz #1, March 28, 2005

In the following problems you are required to show all your work and provide the necessary explanations everywhere to get full credit.

1. Use mathematical induction to prove that

$$4 + 4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + \ldots + 4 \cdot 5^n = 5^{n+1} - 1$$

for any integer $n \geq 0$.

2. Use mathematical induction to prove that $n^{2n+1} \geq (n!)^2$ for any integer $n \geq 1$.

3. Use mathematical induction to prove that $4^{3n} - 1$ is divisible by 63 for any integer $n \geq 0$.

1. Use mathematical induction to prove that

$$4 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + \ldots + 4\cdot 5^n = 5^{n+1} - 1 \qquad (*)$$

for any integer $n \geq 0$.

Proof:

STEP 1: For n=0 $(*)$ is true, since $4 = 5^{0+1} - 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $4+4\cdot 5+4\cdot 5^2+4\cdot 5^3+\ldots+4\cdot 5^k = 5^{k+1}-1$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $4+4\cdot 5+4\cdot 5^2+4\cdot 5^3+\ldots+4\cdot 5^k+4\cdot 5^{k+1} \stackrel{?}{=} 5^{k+2} - 1$. We have

$$4 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + \ldots + 4\cdot 5^k + 4\cdot 5^{k+1} \stackrel{\textbf{ST.2}}{=} 5^{k+1} - 1 + 4\cdot 5^{k+1} \stackrel{?}{=} 5^{k+2} - 1,$$

which is true, since

$$5^{k+1} - 1 + 4\cdot 5^{k+1} \stackrel{?}{=} 5^{k+2} - 1$$
$$\Uparrow$$
$$(1+4)5^{k+1} - 1 \stackrel{?}{=} 5^{k+2} - 1$$
$$\Uparrow$$
$$5\cdot 5^{k+1} - 1 = 5^{k+2} - 1. \blacksquare$$

2. Use mathematical induction to prove that $n^{2n+1} \geq (n!)^2$ for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 this is true, since $1^{2\cdot 1+1} = (1!)^2$.

STEP 2: Suppose $n^{2n+1} \geq (n!)^2$ for some $n = k \geq 1$, that is $k^{2k+1} \geq (k!)^2$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $(k+1)^{2(k+1)+1} \stackrel{?}{\geq} ((k+1)!)^2$. We have

$$((k+1)!)^2 = (k!(k+1))^2 = (k!)^2(k+1)^2 \stackrel{\textbf{ST.2}}{\leq} k^{2k+1}(k+1)^2 \stackrel{?}{\leq} (k+1)^{2(k+1)+1}$$

which is true, since

$$k^{2k+1}(k+1)^2 \stackrel{?}{\leq} (k+1)^{2(k+1)+1}$$
$$\Uparrow$$
$$k^{2k+1}(k+1)^2 \stackrel{?}{\leq} (k+1)^{2k+3}$$
$$\Uparrow$$
$$k^{2k+1} \leq (k+1)^{2k+1}. \blacksquare$$

3. Use mathematical induction to prove that $4^{3n} - 1$ is divisible by 63 for any integer $n \geq 0$.

Proof:

STEP 1: For n=0 this is true, since $4^{3\cdot 0} - 1$ is divisible by 63.

STEP 2: Suppose $4^{3n} - 1$ is true for some $n = k \geq 0$, that is $63 \mid 4^{3k} - 1$.

STEP 3: Prove that $63 \mid 4^{3(k+1)} - 1$. We have

$$4^{3(k+1)} - 1 = 4^{3k+3} - 1 = 4^{3k}\cdot 64 - 1 = 4^{3k}(63+1) - 1 = \underbrace{4^{3k}\cdot 63}_{\text{div. by } 63} + \underbrace{4^{3k} - 1}_{\substack{\text{St. 2} \\ \text{div. by } 63}}. \blacksquare$$