EXAMPLE 1: Prove that
$$3 \mid n^3 - n \qquad (*)$$
for any integer $n \geq 0$.

Proof:

STEP 1: For n=0 $(*)$ is true, since $3 \mid 0^3 - 0$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $3 \mid k^3 - k$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3 \mid (k+1)^3 - (k+1)$. We have
$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = \underbrace{k^3 - k}_{\substack{\text{St. 2}\\ \text{div. by 3}}} + \underbrace{3k^2 + 3k}_{\text{div. by 3}}. \blacksquare$$

EXAMPLE 2: Prove that
$$5 \mid n^5 - n \qquad (*)$$
for any integer $n \geq 0$.

Proof:

STEP 1: For n=0 $(*)$ is true, since $5 \mid 0^5 - 0$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $5 \mid k^5 - k$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $5 \mid (k+1)^5 - (k+1)$. We have
$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = \underbrace{k^5 - k}_{\substack{\text{St. 2}\\ \text{div. by 5}}} + \underbrace{5k^4 + 10k^3 + 10k^2 + 5k}_{\text{div. by 5}}. \blacksquare$$

EXAMPLE 3: Prove that
$$3 \mid n^3 - 7n + 3 \qquad (*)$$
for any integer $n \geq 0$.

Proof:

STEP 1: For n=0 $(*)$ is true, since $3 \mid 0^3 - 7 \cdot 0 + 3$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 0$, that is $3 \mid k^3 - 7k + 3$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3 \mid (k+1)^3 - 7(k+1) + 3$. We have
$$(k+1)^3 - 7(k+1) + 3 = k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 = \underbrace{k^3 - 7k + 3}_{\substack{\text{St. 2}\\ \text{div. by 3}}} + \underbrace{3k^2 + 3k - 6}_{\text{div. by 3}}. \blacksquare$$

EXAMPLE 4: Prove that
$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots\left(1 - \frac{1}{n}\right) = \frac{1}{n} \qquad (*)$$
for any integer $n \geq 2$.

Proof:

STEP 1: For n=2 $(*)$ is true, since $\left(1 - \frac{1}{2}\right) = \frac{1}{2}$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 2$, that is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots\left(1 - \frac{1}{k}\right) = \frac{1}{k}$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1}$. We have:
$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{k+1}\right) \stackrel{\textbf{ST.2}}{=} \frac{1}{k}\left(1 - \frac{1}{k+1}\right) \stackrel{?}{=} \frac{1}{k+1},$$

Approach I: which is true, since

$$\frac{1}{k}\left(1-\frac{1}{k+1}\right)\stackrel{?}{=}\frac{1}{k+1}$$
$$\Uparrow$$
$$\frac{1}{k}\left(\frac{k+1}{k+1}-\frac{1}{k+1}\right)\stackrel{?}{=}\frac{1}{k+1}$$
$$\Uparrow$$
$$\frac{1}{k}\left(\frac{k+1-1}{k+1}\right)\stackrel{?}{=}\frac{1}{k+1}$$
$$\Uparrow$$
$$\frac{1}{k}\left(\frac{k}{k+1}\right)\stackrel{?}{=}\frac{1}{k+1}$$
$$\Uparrow$$
$$\frac{k}{k(k+1)}=\frac{1}{k+1}. \blacksquare$$

Approach II: which is true, since

$$\frac{1}{k}\left(1-\frac{1}{k+1}\right)\stackrel{?}{=}\frac{1}{k+1}$$
$$\Uparrow$$
$$\frac{k}{k}\left(1-\frac{1}{k+1}\right)\stackrel{?}{=}\frac{k}{k+1}$$
$$\Uparrow$$
$$1-\frac{1}{k+1}\stackrel{?}{=}\frac{k}{k+1}$$
$$\Uparrow$$
$$(k+1)-\frac{k+1}{k+1}\stackrel{?}{=}\frac{k(k+1)}{k+1}$$
$$\Uparrow$$
$$k+1-1=k. \blacksquare$$

<u>EXAMPLE 5</u>: Prove that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \qquad (*)$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 $(*)$ is true, since $\frac{1}{1\cdot 3}=\frac{1}{2\cdot 1+1}$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$.

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}\stackrel{?}{=}\frac{k+1}{2(k+1)+1}$. We have

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$
$$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+2-1)(2k+2+1)}$$
$$=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$
$$\stackrel{\mathbf{ST.2}}{=}\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2(k+1)+1},$$

which is true, since

$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2(k+1)+1}$$
$$\Uparrow$$
$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2k+3}$$
$$\Uparrow$$
$$\frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2k+3}$$
$$\Uparrow$$
$$\frac{k(2k+3)+1}{(2k+1)(2k+3)}\stackrel{?}{=}\frac{k+1}{2k+3}$$
$$\Uparrow$$
$$\frac{k(2k+3)+1}{2k+1}\stackrel{?}{=}k+1$$
$$\Uparrow$$
$$k(2k+3)+1\stackrel{?}{=}(k+1)(2k+1)$$
$$\Uparrow$$
$$2k^2+3k+1=2k^2+k+2k+1. \blacksquare$$

<u>EXAMPLE 6</u>: Prove that

$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(n-1)n=\frac{n(n-1)(n+1)}{3} \qquad (*)$$

for any integer $n\geq 2$.

<u>Proof</u>:

STEP 1: For n=2 $(*)$ is true, since $1\cdot 2=\dfrac{2(2-1)(2+1)}{3}$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 2$, that is $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(k-1)k=\dfrac{k(k-1)(k+1)}{3}$.

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(k-1)k+k(k+1)=\dfrac{(k+1)k(k+2)}{3}$.
We have

$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+(k-1)k+k(k+1)\stackrel{\textbf{ST.2}}{=}\frac{k(k-1)(k+1)}{3}+k(k+1)\stackrel{?}{=}\frac{(k+1)k(k+2)}{3},$$

which is true, since

$$\frac{k(k-1)(k+1)}{3}+k(k+1)\stackrel{?}{=}\frac{(k+1)k(k+2)}{3}$$
$$\Uparrow$$
$$k(k-1)(k+1)+3k(k+1)\stackrel{?}{=}(k+1)k(k+2)$$
$$\Uparrow$$
$$(k^2-k)(k+1)+3k^2+3k\stackrel{?}{=}(k^2+k)(k+2)$$
$$\Uparrow$$
$$k^3+k^2-k^2-k+3k^2+3k=k^3+2k^2+k^2+2k. \blacksquare$$

<u>EXAMPLE 7</u>: Prove that

$$2^n<n! \qquad (*)$$

for any integer $n\geq 4$.

<u>Proof</u>:

STEP 1: For n=4 $(*)$ is true, since $2^4<4!$ $(16<24)$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 4$, that is $2^k<k!$.

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is $2^{k+1}\stackrel{?}{<}(k+1)!$. We have

$$2^{k+1}=2\cdot 2^k\stackrel{\textbf{ST.2}}{<}2k!\stackrel{?}{<}(k+1)!,$$

which is true, since

$$2k! \stackrel{?}{<} (k+1)!$$
$$\Uparrow$$
$$2k! \stackrel{?}{<} k!(k+1)$$
$$\Uparrow$$
$$2 \stackrel{?}{<} k+1$$
$$\Uparrow$$
$$1 < k. \blacksquare$$

<u>EXAMPLE 8</u>: Prove that

$$3^n < n! \qquad (*)$$

for any integer $n \geq 7$.

<u>Proof</u>:

STEP 1: For n=7 $(*)$ is true, since $3^7 < 7!$ $(2187 < 5040)$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 7$, that is $3^k < k!$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3^{k+1} \stackrel{?}{<} (k+1)!$. We have

$$3^{k+1} = 3 \cdot 3^k \stackrel{\textbf{ST.2}}{<} 3k! \stackrel{?}{<} (k+1)!,$$

which is true, since

$$3k! \stackrel{?}{<} (k+1)!$$
$$\Uparrow$$
$$3k! \stackrel{?}{<} k!(k+1)$$
$$\Uparrow$$
$$3 \stackrel{?}{<} k+1$$
$$\Uparrow$$
$$2 < k. \blacksquare$$

<u>EXAMPLE 9</u>: Prove that

$$3^n \geq 2n+1 \qquad (*)$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 $(*)$ is true, since $3^1 = 2 \cdot 1 + 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $3^k \geq 2k+1$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3^{k+1} \geq 2(k+1)+1$. We have

$$3^{k+1} = 3 \cdot 3^k \stackrel{\textbf{ST.2}}{\geq} 3(2k+1) \stackrel{?}{\geq} 2(k+1)+1,$$

which is true, since

$$3(2k+1) \stackrel{?}{\geq} 2(k+1)+1$$
$$\Uparrow$$
$$6k+3 \stackrel{?}{\geq} 2k+2+1$$
$$\Uparrow$$
$$4k \geq 0. \blacksquare$$

<u>EXAMPLE 10</u>: Prove that

$$2^{n+2} \geq 2n+5 \qquad (*)$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 $(*)$ is true, since $2^{1+2} \geq 2 \cdot 1 + 5$.
STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $2^{k+2} \geq 2k + 5$.
STEP 3: Prove that $(*)$ is true for $n = k + 1$, that is $2^{k+3} \geq 2(k + 1) + 5$. We have

$$2^{k+3} = 2 \cdot 2^{k+2} \overset{\mathbf{ST.2}}{\geq} 3(2k + 5) \overset{?}{\geq} 2(k + 1) + 5,$$

which is true, since

$$\begin{gathered} 3(2k + 5) \overset{?}{\geq} 2(k + 1) + 5 \\ \Uparrow \\ 6k + 15 \overset{?}{\geq} 2k + 2 + 5 \\ \Uparrow \\ 4k + 8 \geq 0. \blacksquare \end{gathered}$$