EXAMPLE 1: Prove that

$$3 \mid 4^n - 1 \qquad (*)$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 $(*)$ is true, since $3 \mid 4^1 - 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $3 \mid 4^k - 1$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $4 \mid 4^{k+1} - 1$. We have

$$4^{k+1} - 1 = 4 \cdot 4^k - 1 = (3+1)4^k - 1 = \underbrace{3 \cdot 4^k}_{\text{div. by } 3} + \underbrace{4^k - 1}_{\substack{\text{St. 2} \\ \text{div. by } 3}}. \blacksquare$$

EXAMPLE 2: Prove that

$$8 \mid 3^{2n} - 1 \qquad (*)$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 $(*)$ is true, since $8 \mid 3^{2 \cdot 1} - 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $8 \mid 3^{2k} - 1$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $8 \mid 3^{2(k+1)} - 1$. We have

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 9 - 1 = 3^{2k}(8+1) - 1 = \underbrace{3^{2k} \cdot 8}_{\text{div. by } 8} + \underbrace{3^{2k} - 1}_{\substack{\text{St. 2} \\ \text{div. by } 8}}. \blacksquare$$

EXAMPLE 3: Prove that

$$7 \mid n^7 - n \qquad (*)$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 $(*)$ is true, since $7 \mid 1^7 - 1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $7 \mid k^7 - k$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $7 \mid (k+1)^7 - (k+1)$. We have

$$(k+1)^7 - (k+1) = k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1$$

$$= \underbrace{k^7 - k}_{\substack{\text{St. 2} \\ \text{div. by } 7}} + \underbrace{7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k}_{\text{div. by } 7}. \blacksquare$$

EXAMPLE 4: Prove that

$$1 + 3 + 5 + \ldots + (2n-1) = n^2 \qquad (*)$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 $(*)$ is true, since $1 = 1^2$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $1 + 3 + 5 + \ldots + (2k-1) = k^2$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $1 + 3 + 5 + \ldots + (2k-1) + (2k+1) \stackrel{?}{=} (k+1)^2$. We have:

$$1 + 3 + 5 + \ldots + (2k-1) + (2k+1) \stackrel{\mathbf{ST.2}}{=} k^2 + (2k+1) = (k+1)^2. \blacksquare$$

<u>EXAMPLE 5</u>: Prove that

$$1+2+3+\ldots+n=\frac{n(n+1)}{2} \tag{$*$}$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 $(*)$ is true, since $1=\dfrac{1(1+1)}{2}$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is $1+2+3+\ldots+k=\dfrac{k(k+1)}{2}$.

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is $1+2+3+\ldots+k+(k+1)\stackrel{?}{=}\dfrac{(k+1)(k+2)}{2}$. We have

$$1+2+3+\ldots+k+(k+1)\stackrel{\textbf{ST.2}}{=}\frac{k(k+1)}{2}+(k+1)\stackrel{?}{=}\frac{(k+1)(k+2)}{2},$$

which is true, since

$$\frac{k(k+1)}{2}+(k+1)\stackrel{?}{=}\frac{(k+1)(k+2)}{2}$$
$$\Uparrow$$
$$k(k+1)+2(k+1)\stackrel{?}{=}(k+1)(k+2)$$
$$\Uparrow$$
$$k^2+k+2k+2=k^2+3k+2. \blacksquare$$

<u>EXAMPLE 6</u>: Prove that

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6} \tag{$*$}$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 $(*)$ is true, since $1^2=\dfrac{1(1+1)(2\cdot 1+1)}{6}$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is $1^2+2^2+3^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$.

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is $1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\dfrac{(k+1)(k+2)(2k+3)}{6}$. We have

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2\stackrel{\textbf{ST.2}}{=}\frac{k(k+1)(2k+1)}{6}+(k+1)^2\stackrel{?}{=}\frac{(k+1)(k+2)(2k+3)}{6},$$

which is true, since

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2\stackrel{?}{=}\frac{(k+1)(k+2)(2k+3)}{6}$$
$$\Uparrow$$
$$k(k+1)(2k+1)+6(k+1)^2\stackrel{?}{=}(k+1)(k+2)(2k+3)$$
$$\Uparrow$$
$$(k^2+k)(2k+1)+6(k+1)^2\stackrel{?}{=}(k^2+3k+2)(2k+3)$$
$$\Uparrow$$
$$2k^3+k^2+2k^2+k+6k^2+12k+6=2k^3+3k^2+6k^2+9k+4k+6. \blacksquare$$

<u>EXAMPLE 7</u>: Prove that

$$2^n>n \tag{$*$}$$

for any integer $n \geq 1$.

<u>Proof</u>:

STEP 1: For n=1 $(*)$ is true, since $2^1>1$.

STEP 2: Suppose $(*)$ is true for some $n=k\geq 1$, that is $2^k>k$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $2^{k+1} \overset{?}{>} k+1$. We have

$$2^{k+1} = 2 \cdot 2^k \overset{\textbf{ST.2}}{>} 2k \overset{?}{\geq} k+1,$$

which is true, since $k \geq 1$. ■

EXAMPLE 8: Prove that

$$n! \leq n^n \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

STEP 1: For n=1 $(*)$ is true, since $1! = 1^1$.

STEP 2: Suppose $(*)$ is true for some $n = k \geq 1$, that is $k! \leq k^k$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $(k+1)! \overset{?}{\leq} (k+1)^{k+1}$. We have

$$(k+1)! = k! \cdot (k+1) \overset{\textbf{ST.2}}{\leq} k^k \cdot (k+1) \overset{?}{\leq} (k+1)^{k+1},$$

which is true, since $k^k < (k+1)^k$. ■

I. Prove by induction the following identities:

1. $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots\left(1 - \frac{1}{n}\right) = \frac{1}{n}.$

2. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$

3. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1)n = \frac{n(n-1)(n+1)}{3}.$

4*. $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{n-1}+\sqrt{n}} = \sqrt{n} - 1.$

5**. $\sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots + \sin n\varphi = \dfrac{\sin\frac{n\varphi}{2}\sin\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}.$

II. Prove by induction the following inequalities:

1. $2^n < n!$ for any integer $n \geq 4$.

2. $3^n < n!$ for any integer $n \geq 7$.

3. $3^n \geq 2n+1$ for any integer $n \geq 1$.

4. $2^{n+2} \geq 2n+5$ for any integer $n \geq 1$.

5*. $(2n)! < 2^{2n}(n!)^2$ for any integer $n \geq 1$.

6**. $(n+1)^n < n^{n+1}$ for any integer $n \geq 3$.

7**. $\frac{a_1^n + a_2^n}{2} \geq \left(\frac{a_1+a_2}{2}\right)^n$ for any positive numbers a_1, a_2 and for any integer $n \geq 1$.

8***. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$ for any integer $n \geq 1$.

III. Prove by induction the following problems:

1. $n^3 - n$ is divisible by 3 for any nonnegative integer n.

2. $n^5 - n$ is divisible by 5 for any nonnegative integer n.

3. $n^3 - 7n + 3$ is divisible by 3 for any nonnegative integer n.

4. $4^n - 1$ is divisible by 3 for any nonnegative integer n.

5. $3^{2n} - 1$ is divisible by 8 for any positive integer n.

6*. $3^{2n+3} + 40n - 27$ is divisible by 64 for any nonnegative integer n.

7*. $5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19 for any nonnegative integer n.

8**. $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ is integer for any nonnegative integer n.