GRAPH THEORY: INTRODUCTION

DEFINITION 1:

A **graph** G consists of two finite sets: a set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**. An edge with just one endpoint is called a **loop**, and two distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**. An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**. A graph with no vertices is called **empty**, and one with at least one vertex is called **nonempty**.

EXAMPLE:

Consider the following graph:

(a) Write the vertex set and the edge set, and give a table showing the edge-endpoint function;
(b) Find all that are incident on v_1, all vertices that are adjacent to v_1, all edges that are adjacent to e_1, all loops, all edges, all vertices that are adjacent to themselves, all isolated vertices.

Solution:

(a) We have: vertex set $= \{v_1, v_2, v_3, v_4, v_5, v_6\}$
edge set $= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
edge-endpoint function :

Edges	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

(b) We have: , , and are incident on v_1.

and are adjacent to v_1.

, , and are adjacent to e_1.

and are loops.

and are parallel.

and are adjacent to themselves.

is an isolated vertex.

DEFINITION 2:

A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{v, w\}$.

EXAMPLE:

Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$.

Solution:

There are 5 such graphs:

DEFINITION 3:

A **complete graph on n vertices**, denoted K_n, is a simple graph with n vertices $v_1, v_2, \ldots, v_n$ whose set of edges contains exactly one edge for each pair of distinct vertices.

EXAMPLE:

Draw the complete graphs K_2, K_3, K_4, and K_5.

Solution:

DEFINITION 4:

A **complete bipartite graph on (m, n) vertices**, denoted $\boldsymbol{K_{m,n}}$, is a simple graph with vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties:

for all $i, k = 1, 2, \ldots, m$ and all $j, l = 1, 2, \ldots, n$,

1. There is an edge from each vertex v_i to each vertex w_j;
2. There is not an edge from any vertex v_i to any other vertex v_k;
3. There is not an edge from any vertex w_j to any other vertex w_l.

EXAMPLE:

Draw the bipartite graphs $K_{3,2}$ and $K_{3,3}$.

Solution:

DEFINITION 5:

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as in G.

EXAMPLE:

List all nonempty subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2, and e_3 is a loop at v_1.

Solution:

We first draw the graph:

This graph has 11 subgraphs:

DEFINITION 6:

Let G be a graph and v a vertex of G. The **degree of v,** denoted **deg(v)**, equals the number of edges that are incident on v, with an edge that is a loop counted twice. The **total degree of G** is the sum of the degrees of all the vertices of G.

EXAMPLE:

Find the degree of each vertex of the graph G shown below. Then find the total degree of G.

Solution:

THEOREM:

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

COROLLARY 1:

The total degree of a graph is even.

EXAMPLE:

Draw a graph with the specified properties or show that no such graph exists.
(a) Graph with four vertices of degrees 1, 1, 2, and 3.
(b) Graph with four vertices of degrees 1, 1, 3, and 3.
(c) Simple graph with four vertices of degrees 1, 1, 3, and 3.

Solution:

<u>COROLLARY 2</u>:

In any graph there are an even number of vertices of odd degree.

<u>PROBLEM</u>: Is it possible in a group of 9 people for each to shake hands with exactly 5 other persons?

<u>Solution</u>: The answer is no. In fact, imagine a graph in which each of the 9 people is represented by a dot and two dots are joined by an edge if, and only if, the people they represent shook hands. Suppose each of the people shook hands with exactly 5 others. Then we have an odd number (nine) vertices of odd degree. This contradicts Corollary 2. ■