The Pigeonhole Principle

<u>PRINCIPLE</u>:

If we put $N + 1$ or more pigeons into N pigeon holes, then at least one pigeon hole will contain two or more pigeons.

<u>FIRST SIMPLE EXAMPLES</u>:

1. *Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.*

2. *Among a group of 11 people in the elevator of a 10 story building, there must be at least two who will exit the elevator on the same floor.*

3. *One million trees grow in a forest. It is known that no tree has more than 600,000 leaves. Show that at any moment there are at least two trees in the forest that have exactly the same number of leaves.*

<u>Solution</u>: Trees are pigeons and numbers of their leaves are pigeon holes. As there are more pigeons than pigeon holes, there will be a pigeon hole with more than one pigeon in it.

<u>FURTHER EXAMPLES</u>:

1. *12 students wrote a dictation. John Smart made 10 errors, each of the other students made less than that number. Prove that at least two students made equal number of errors.*

<u>Solution</u>: Let us pretend that the students are 'pigeons' and put them in 11 'holes' numbered 0, 1, 2, ... , 10, according to the number of errors made. In hole 0 we put those students who made no errors, in hole 1 those who made exactly 1 error, in hole 2 those who made 2 errors, and so on. Certainly, hole 10 is occupied solely by John Smart. Now apply the Pigeonhole Principle.

2. *How many cards must be selected from a standard deck of 52 cards to ensure that we get at least 3 cards of the same suit?*

<u>Solution</u>: Since there are 4 suits, if we only select 8 cards then it is possible that we get 2 cards of each suit. So 8 is not enough to guarantee at least 3 cards of the same suit. However, if we select 9 cards then the Pigeonhole Principle tells us that we will get at least $9/4 = 3$ cards of the same suit. So 9 is the least we can select to guarantee at least 3 cards of the same suit.

3. *Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Prove that if five integers are selected from A, then at least one pair of integers have a sum of 9.*

Solution: Partition the set A into 4 subsets:

$$\{1, 8\}, \quad \{2, 7\}, \quad \{3, 6\}, \quad \text{and} \quad \{4, 5\},$$

each consisting of two integers whose sum is 9. If 5 integers are selected from A, then by the Pigeonhole Principle at least two must be from the same subset. But then the sum of these two integers is 9.

4. *A plane is colored blue and red. Is it always possible to find two points of the same color exactly 1 inch apart?*

Solution: Think of an equilateral triangle with each side exactly 1 inch long. At least two of its vertices have to be of the same color (colors are pigeon holes, and vertices of the triangle are pigeons). This proves that there have to be two points of the same color exactly 1 inch apart.

5. *51 points were placed, in an arbitrary way, into the square of side 1. Prove that some 3 of these points can be covered by a circle of radius 1/7 .*

Solution: Divide the square into 25 smaller squares of side 1/5 each. Then at least one of these small squares - 'holes'- would contain at least three 'pigeons' - points. Indeed, if this is not true, then every small square contains 2 points or less; but the the total number of points is no more that $2 \cdot 25 = 50$. This contradicts to the assumption that we have 51 points.

Now the circle circumvented around the square with the three points inside also contains these three points and has radius

$$r = \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \sqrt{\frac{2}{100}} = \sqrt{\frac{1}{50}} < \sqrt{\frac{1}{49}} = \frac{1}{7}.$$

PROBLEMS:

1. Population of Greater Manchester is above 6,000,000 people, and each has no more than 100,000 hairs on his or her head. Prove that some 60 residents of Greater Manchester have equal number of hairs.

2. There are 30 classes and 1000 students in the school. Prove that at least one class has at least 34 students.

3. A group of 25 students wrote a dictation. John Smart made 10 errors, and each of the rest made less than 10 errors. Prove that at least 3 students made equal number of errors.

4. Prove that, given any 12 natural numbers, we can chose two of them and such that their difference is divisible by 11.

5*. 5 points are positioned inside of the equilateral triangle of side 2. Prove that there are two of them at the distance less than 1 from each other.

6*. Prove that of any 52 natural numbers one can find two numbers m and n such that either their sum $m + n$ or their difference $m - n$ is divisible by 100. Is the same statement true for 51 arbitrary natural numbers?

7.** Prove that some integral power of 2 has the decimal expansion which starts with the digits 1999:

$$2^n = 1999...$$

3. *A group of 25 students wrote a dictation. John Smart made 10 errors, and each of the rest made less than 10 errors. Prove that at least 3 students made equal number of errors.*

Solution: Assume that no three students made equal number of errors. It means that each of 10 holes 0, 1, 2, 3, ... , 9 contains less than three students. Therefore all these holes together contain less than $10 \cdot 2 = 20$ students. Add John Smart to this number, and we get only 21 students, not 25 as given in the problem. We reached a contradiction.

4. *Prove that, given any 12 natural numbers, we can chose two of them and such that their difference is divisible by 11.*

Solution: There are 11 possible remainders upon division by 11:

$$0,\ 1,\ 2,\ 3, \ldots, 10.$$

But we have 12 numbers. if we take the remainders for 'holes' and the numbers for 'pigeons' then by the Pigeonhole Principle there are at least two pigeons sharing the same hole, i.e. two numbers with the same remainder. The difference of these two numbers is divisible by 11.