COUNTING SUBSETS OF A SET: COMBINATIONS

DEFINITION 1:

Let n, r be nonnegative integers with $r \leq n$. An r-combination of a set of n elements is a subset of r of the n elements.

EXAMPLE 1: Let $S = \{a, b, c, d\}$. Then

$$\text{the 1-combinations are} : \{a\},\ \{b\},\ \{c\},\ \{d\}$$

$$\text{the 2-combinations are} : \{a, b\},\ \{a, c\},\ \{a, d\},\ \{b, c\},\ \{b, d\},\ \{c, d\}$$

$$\text{the 3-combinations are} : \{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\}$$

$$\text{the 4-combination is} : \{a, b, c, d\}.$$

DEFINITION 2:

The symbol $\binom{n}{r}$, read "n choose r," denotes the number of r-combinations that can be chosen from a set of n elements.

EXAMPLE 2: It follows from Example 1 that

$$\binom{4}{1} = 4, \quad \binom{4}{2} = 6, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1.$$

THEOREM: Let n, r be nonnegative integers with $r \leq n$. Then

$$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}.$$

EXAMPLE 3: We have

$$\binom{4}{2} = \frac{4!}{2! \cdot (4-2)!} = \frac{4!}{2! \cdot 2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2) \cdot (1 \cdot 2)} = \frac{3 \cdot 4}{1 \cdot 2} = 6,$$

$$\binom{4}{3} = \frac{4!}{3! \cdot (4-3)!} = \frac{4!}{3! \cdot 1!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2 \cdot 3) \cdot 1} = \frac{4}{1} = 4,$$

$$\binom{4}{4} = \frac{4!}{4! \cdot (4-4)!} = \frac{4!}{4! \cdot 0!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2 \cdot 3 \cdot 4) \cdot 1} = \frac{1}{1} = 1,$$

$$\binom{8}{5} = \frac{8!}{5! \cdot (8-5)!} = \frac{8!}{5! \cdot 3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) \cdot (1 \cdot 2 \cdot 3)} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3} = \frac{7 \cdot 8}{1} = 56.$$

PROBLEMS:

1. Suppose 5 members of a group of 12 are to be chosen to work as a team on a special project. How many distinct 5-person teams can be formed?

2. Suppose two members of the group of 12 insist on working as a pair — any team must either contain both or neither. How many distinct 5-person teams can be formed?

3. Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?

4. Suppose the group of 12 consists of 5 men and 7 women.
(a) How many 5-person teams can be chosen that consist of 3 men and 2 women?
(b) How many 5-person teams contain at least one man?
(c) How many 5-person teams contain at most one man?

5. Consider various ways of ordering the letters in the word MISSISSIPPI:

IIMSSPISSIP, ISSSPMIIPIS, PIMISSSSIIP, and so on.

How many distinguishable orderings are there?

SOLUTIONS:

1. *Suppose 5 members of a group of 12 are to be chosen to work as a team on a special project. How many distinct 5-person teams can be formed?*

Solution: The number of distinct 5-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of 12. This number is

$$\binom{12}{5} = \frac{12!}{5!\cdot(12-5)!} = \frac{12!}{5!\cdot 7!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{(1\cdot 2\cdot 3\cdot 4\cdot 5)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)}$$

$$= \frac{8\cdot 9\cdot 10\cdot 11\cdot 12}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{9\cdot 10\cdot 11\cdot 12}{1\cdot 3\cdot 5} = 3\cdot 2\cdot 11\cdot 12 = 792. \blacksquare$$

2. *Suppose two members of the group of 12 insist on working as a pair — any team must either contain both or neither. How many distinct 5-person teams can be formed?*

Solution: Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B. By Theorem 1 (The Addition Rule) we have:

$$\begin{bmatrix}\text{number of 5-person teams}\\ \text{containing both } A \text{ and } B\\ \text{or neither } A \text{ nor } B\end{bmatrix} = \begin{bmatrix}\text{number of 5-person}\\ \text{teams containing}\\ \text{both } A \text{ and } B\end{bmatrix} + \begin{bmatrix}\text{number of 5-person}\\ \text{teams containing}\\ \text{neither } A \text{ nor } B\end{bmatrix}.$$

Because a team that contains both A and B contains exactly 3 other people from the remaining 10 in the group, there are as many such teams as there are subsets of 3 people that can be chosen from the remaining 10. This number is

$$\binom{10}{3} = \frac{10!}{3!\cdot(10-3)!} = \frac{10!}{3!\cdot 7!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{(1\cdot 2\cdot 3)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7)} = \frac{8\cdot 9\cdot 10}{1\cdot 2\cdot 3} = 4\cdot 3\cdot 10 = 120.$$

Similarly, because a team that contains neither A nor B contains exactly 5 people from the remaining 10 in the group, there are as many such teams as there are subsets of 5 people that can be chosen from the remaining 10. This number is

$$\binom{10}{5} = \frac{10!}{5!\cdot(10-5)!} = \frac{10!}{5!\cdot 5!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{(1\cdot 2\cdot 3\cdot 4\cdot 5)\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5)} = \frac{6\cdot 7\cdot 8\cdot 9\cdot 10}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 252.$$

Therefore,

$$\begin{bmatrix}\text{number of 5-person teams}\\ \text{containing both } A \text{ and } B\\ \text{or neither } A \text{ nor } B\end{bmatrix} = 120 + 252 = 372. \blacksquare$$

3. *Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?*

Solution: Call the two members of the group that refuse to work together A and B. By Theorem 2 (The Difference Rule) we have:

$$\begin{bmatrix} \text{number of 5-person teams} \\ \text{that don't contain} \\ \text{both } A \text{ and } B \end{bmatrix} = \begin{bmatrix} \text{total number of} \\ \text{5-person teams} \end{bmatrix} - \begin{bmatrix} \text{number of 5-person teams} \\ \text{that contain} \\ \text{both } A \text{ and } B \end{bmatrix}$$

$$= \binom{12}{5} - \binom{10}{3} = 792 - 120 = 672. \blacksquare$$

4. *Suppose the group of 12 consists of 5 men and 7 women.*
(a) How many 5-person teams can be chosen that consist of 3 men and 2 women?
(b) How many 5-person teams contain at least one man?
(c) How many 5-person teams contain at most one man?

Solution:

(a) Note, that there are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven. Therefore, by The Multiplication Rule we have:

$$\begin{bmatrix} \text{number of 5-person teams that} \\ \text{contain 3 men and 2 women} \end{bmatrix} = \binom{5}{3} \cdot \binom{7}{2} = 210.$$

(b) By Theorem 2 (The Difference Rule) we have:

$$\begin{bmatrix} \text{number of 5-person teams} \\ \text{with at least one man} \end{bmatrix} = \begin{bmatrix} \text{total number} \\ \text{of 5-person teams} \end{bmatrix} - \begin{bmatrix} \text{number of 5-person teams} \\ \text{that do not contain any men} \end{bmatrix}.$$

Now a 5-person team with no men consists of 5 women chosen from the seven women in the group. So, there are $\binom{7}{5} = 21$ such teams. Also, the total number of 5-person teams is $\binom{12}{5} = 792$. Therefore,

$$\begin{bmatrix} \text{number of 5-person teams} \\ \text{with at least one man} \end{bmatrix} = 792 - 21 = 771.$$

(c) By Theorem 1 (The Addition Rule) we have:

$$\begin{bmatrix} \text{number of 5-person teams} \\ \text{with at most one man} \end{bmatrix} = \begin{bmatrix} \text{number of 5-person teams} \\ \text{without any men} \end{bmatrix} + \begin{bmatrix} \text{number of 5-person teams} \\ \text{with one man} \end{bmatrix}.$$

Now a 5-person team without any men consists of 5 women chosen from the 7 women in the group. So, there are $\binom{7}{5} = 21$ such teams. Also, by The Multiplication Rule there are $\binom{5}{1} \cdot \binom{7}{4} = 175$ teams with one man. Therefore,

$$\begin{bmatrix} \text{number of 5-person teams} \\ \text{with at most one man} \end{bmatrix} = 21 + 175 = 196. \blacksquare$$

5. *Consider various ways of ordering the letters in the word MISSISSIPPI:*

$$IIMSSPISSIP, \; ISSSPMIIPIS, \; PIMISSSSIIP, \quad \textit{and so on.}$$

How many distinguishable orderings are there?

Solution: Since there are 11 positions in all, there are

$$\binom{11}{4} \text{ subsets of 4 positions for the } S\text{'s.}$$

Once the four S's are in place, there are

$$\binom{7}{4} \text{ subsets of 4 positions for the } I\text{'s.}$$

After the I's are in place, there are

$$\binom{3}{2} \text{ subsets of 2 positions for the } P\text{'s.}$$

That leaves just one position for the M. Hence, by The Multiplication Rule we have:

$$\begin{bmatrix} \text{number of ways to} \\ \text{position all the letters} \end{bmatrix} = \binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} = 34,650. \blacksquare$$