COUNTING SUBSETS OF A SET: COMBINATIONS

DEFINITION 1:
Let \( n, r \) be nonnegative integers with \( r \leq n \). An \( r \)-combination of a set of \( n \) elements is a subset of \( r \) of the \( n \) elements.

EXAMPLE 1: Let \( S = \{a, b, c, d\} \). Then
the 1-combinations are: \( \{a\}, \{b\}, \{c\}, \{d\} \)
the 2-combinations are: \( \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \)
the 3-combinations are: \( \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \)
the 4-combination is: \( \{a, b, c, d\} \).

DEFINITION 2:
The symbol \( \binom{n}{r} \), read "\( n \) choose \( r \)," denotes the number of \( r \)-combinations that can be chosen from a set of \( n \) elements.

EXAMPLE 2: It follows from Example 1 that
\[
\binom{4}{1} = 4, \quad \binom{4}{2} = 6, \quad \binom{4}{3} = 4, \quad \binom{4}{4} = 1.
\]

THEOREM: Let \( n, r \) be nonnegative integers with \( r \leq n \). Then
\[
\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}.
\]

EXAMPLE 3: We have
\[
\binom{4}{2} = \frac{4!}{2! \cdot (4-2)!} = \frac{4!}{2! \cdot 2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2) \cdot (1 \cdot 2)} = \frac{3 \cdot 4}{1 \cdot 2} = 6,
\]
\[
\binom{4}{3} = \frac{4!}{3! \cdot (4-3)!} = \frac{4!}{3! \cdot 1!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2 \cdot 3) \cdot 1} = \frac{4}{1} = 4,
\]
\[
\binom{4}{4} = \frac{4!}{4! \cdot (4-4)!} = \frac{4!}{4! \cdot 0!} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(1 \cdot 2 \cdot 3 \cdot 4) \cdot 1} = \frac{1}{1} = 1,
\]
\[
\binom{8}{5} = \frac{8!}{5! \cdot (8-5)!} = \frac{8!}{5! \cdot 3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) \cdot (1 \cdot 2 \cdot 3)} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3} = \frac{7 \cdot 8}{1} = 56.
\]
PROBLEMS:

1. Suppose 5 members of a group of 12 are to be chosen to work as a team on a special project. How many distinct 5-person teams can be formed?

2. Suppose two members of the group of 12 insist on working as a pair — any team must either contain both or neither. How many distinct 5-person teams can be formed?

3. Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?

4. Suppose the group of 12 consists of 5 men and 7 women.
   (a) How many 5-person teams can be chosen that consist of 3 men and 2 women?
   (b) How many 5-person teams contain at least one man?
   (c) How many 5-person teams contain at most one man?

5. Consider various ways of ordering the letters in the word MISSISSIPPI:

   IIMSSPIISSIP, ISSSPMIIPIS, PIMISSSSIIP, and so on.

   How many distinguishable orderings are there?
1. Suppose 5 members of a group of 12 are to be chosen to work as a team on a special project. How many distinct 5-person teams can be formed?

Solution: The number of distinct 5-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of 12. This number is

\[
\binom{12}{5} = \frac{12!}{5! \cdot (12 - 5)!} = \frac{12!}{5! \cdot 7!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 3 \cdot 2 \cdot 11 \cdot 12 = 792. \]

2. Suppose two members of the group of 12 insist on working as a pair — any team must either contain both or neither. How many distinct 5-person teams can be formed?

Solution: Call the two members of the group that insist on working as a pair A and B. Then any team formed must contain both A and B or neither A nor B. By Theorem 1 (The Addition Rule) we have:

\[
\begin{bmatrix}
\text{number of 5-person teams} \\
\text{containing both A and B} \\
\text{or neither A nor B}
\end{bmatrix} = \begin{bmatrix}
\text{number of 5-person teams containing} \\
\text{both A and B} \\
\text{neither A nor B}
\end{bmatrix} + \begin{bmatrix}
\text{number of 5-person teams containing} \\
\text{both A and B} \\
\text{neither A nor B}
\end{bmatrix}.
\]

Because a team that contains both A and B contains exactly 3 other people from the remaining 10 in the group, there are as many such teams as there are subsets of 3 people that can be chosen from the remaining 10. This number is

\[
\binom{10}{3} = \frac{10!}{3! \cdot (10 - 3)!} = \frac{10!}{3! \cdot 7!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3} = \frac{8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3} = 4 \cdot 3 \cdot 10 = 120.
\]

Similarly, because a team that contains neither A nor B contains exactly 5 people from the remaining 10 in the group, there are as many such teams as there are subsets of 5 people that can be chosen from the remaining 10. This number is

\[
\binom{10}{5} = \frac{10!}{5! \cdot (10 - 5)!} = \frac{10!}{5! \cdot 5!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.
\]

Therefore,

\[
\begin{bmatrix}
\text{number of 5-person teams} \\
\text{containing both A and B} \\
\text{or neither A nor B}
\end{bmatrix} = 120 + 252 = 372. \]
3. Suppose two members of the group of 12 refuse to work together on a team. How many distinct 5-person teams can be formed?

Solution: Call the two members of the group that refuse to work together A and B. By Theorem 2 (The Difference Rule) we have:

\[
\begin{bmatrix}
\text{number of 5-person teams that don't contain both A and B}
\end{bmatrix} = \begin{bmatrix}
\text{total number of 5-person teams}
\end{bmatrix} - \begin{bmatrix}
\text{number of 5-person teams that contain both A and B}
\end{bmatrix}
\]

\[
= \binom{12}{5} - \binom{10}{3} = 792 - 120 = 672.
\]

4. Suppose the group of 12 consists of 5 men and 7 women.

(a) How many 5-person teams can be chosen that consist of 3 men and 2 women?

(b) How many 5-person teams contain at least one man?

(c) How many 5-person teams contain at most one man?

Solution:

(a) Note, that there are \(\binom{5}{3}\) ways to choose the three men out of the five and \(\binom{7}{2}\) ways to choose the two women out of the seven. Therefore, by The Multiplication Rule we have:

\[
\begin{bmatrix}
\text{number of 5-person teams that contain 3 men and 2 women}
\end{bmatrix} = \binom{5}{3} \cdot \binom{7}{2} = 210.
\]

(b) By Theorem 2 (The Difference Rule) we have:

\[
\begin{bmatrix}
\text{number of 5-person teams with at least one man}
\end{bmatrix} = \begin{bmatrix}
\text{total number of 5-person teams}
\end{bmatrix} - \begin{bmatrix}
\text{number of 5-person teams that do not contain any men}
\end{bmatrix}.
\]

Now a 5-person team with no men consists of 5 women chosen from the seven women in the group. So, there are \(\binom{7}{5} = 21\) such teams. Also, the total number of 5-person teams is \(\binom{12}{5} = 792\). Therefore,

\[
\begin{bmatrix}
\text{number of 5-person teams with at least one man}
\end{bmatrix} = 792 - 21 = 771.
\]

(c) By Theorem 1 (The Addition Rule) we have:

\[
\begin{bmatrix}
\text{number of 5-person teams with at most one man}
\end{bmatrix} = \begin{bmatrix}
\text{number of 5-person teams without any men}
\end{bmatrix} + \begin{bmatrix}
\text{number of 5-person teams with one man}
\end{bmatrix}.
\]

Now a 5-person team without any men consists of 5 women chosen from the 7 women in the group. So, there are \(\binom{7}{5} = 21\) such teams. Also, by The Multiplication Rule there are \(\binom{5}{1} \cdot \binom{7}{4} = 175\) teams with one man. Therefore,

\[
\begin{bmatrix}
\text{number of 5-person teams with at most one man}
\end{bmatrix} = 21 + 175 = 196.
\]
5. Consider various ways of ordering the letters in the word MISSISSIPPI:

\text{IIMSSPISSIP, ISSPMIIPIS, PIMISSSIIP, and so on.}

How many distinguishable orderings are there?

Solution: Since there are 11 positions in all, there are

\( \binom{11}{4} \) subsets of 4 positions for the S’s.

Once the four S’s are in place, there are

\( \binom{7}{4} \) subsets of 4 positions for the I’s.

After the I’s are in place, there are

\( \binom{3}{2} \) subsets of 2 positions for the P’s.

That leaves just one position for the M. Hence, by The Multiplication Rule we have:

\[
\begin{align*}
\text{number of ways to position all the letters} &= \binom{11}{4} \cdot \binom{7}{4} \cdot \binom{3}{2} \\
&= 34,650. ■
\end{align*}
\]