COUNTING ELEMENTS OF SETS

NOTATION: For any finite set, $n(A)$ denotes the number of elements in A.

THEOREM 1 (The Addition Rule):
Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then

$$n(A) = n(A_1) + n(A_2) + \ldots + n(A_k).$$

THEOREM 2 (The Difference Rule):
If A is a finite set and B is a subset of A, then

$$n(A - B) = n(A) - n(B).$$

THEOREM 3 (The Inclusion/Exclusion Rule):
If $A, B,$ and C are any finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and

$$n(A \cup B \cup C)$$
$$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

EXAMPLES:

1. *A code word consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different code words are possible?*

Solution: By Theorem 1, the total number of code words equals the

number of code words of length 1 (which $= 26$)
+
number of code words of length 2 (which $= 26^2$)
+
number of code words of length 3 (which $= 26^3$).

Hence, the total number of code words

$$= 26 + 26^2 + 26^3 = 18,278. \blacksquare$$

2. *How many integers from 1 through 999 do not have any repeated digits?*
Solution: By Theorem 1, the number of integers from 1 through 999 with no repeated digits equals the

number of integers from 1 through 9 with no repeated digits (which $= 9$)
+
number of integers from 10 through 99 with no repeated digits (which $= 9 \cdot 9$)
+
number of integers from 100 through 999 with no repeated digits (which $= 9 \cdot 9 \cdot 8$).

Hence, the total number of integers from 1 through 999 with no repeated digits

$$= 9 + 9 \cdot 9 + 9 \cdot 9 \cdot 8 = 738. \blacksquare$$

3. *How many integers from 1 through 999 have at least one repeated digit?*
Solution: By Theorem 2, the number of integers from 1 through 999 with at least one repeated digit equals the

total number integers from 1 through 999 (which $= 999$)
-
number of integers from 1 through 999 with no repeated digits (which $= 738$).

Hence, the total number of integers from 1 through 999 with at least one repeated digit

$$= 999 - 738 = 261. \blacksquare$$

4. *How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?*
Solution: We will use Theorem 3. Let

$A =$ the set of all integers from 1 through 1,000 that are multiples of 3
$B =$ the set of all integers from 1 through 1,000 that are multiples of 5.

Then

$A \cup B =$ the set of all integers from 1 through 1,000 that are multiples of 3 or 5

$A \cap B =$ the set of all integers from 1 through 1,000 that are multiples of 3 and 5
$\quad\;\; =$ the set of all integers from 1 through 1,000 that are multiples of 15.

By Theorem 3,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

So, to solve this problem we should find $n(A)$, $n(B)$, and $n(A \cap B)$. One can see that
$\boldsymbol{n(A)}$: there are $[1,000/3] = 333$ numbers from 1 through 1,000 that are multiples of 3
$\boldsymbol{n(B)}$: there are $[1,000/5] = 200$ numbers from 1 through 1,000 that are multiples of 5
$\boldsymbol{n(A \cap B)}$: there are $[1,000/15] = 66$ numbers from 1 through 1,000 that are multiples of 15.
Hence by Theorem 3, the total number of integers from 1 through 1,000 that are multiples of 3 or multiples of 5

$$= 333 + 200 - 66 = 467. \blacksquare$$

EXERCISES:

1. How many arrangements of no more that three letters can be formed using the letters of the word $NETWORK$ with

(a) repetitions allowed?

(b) no repetitions allowed?

2. (a) How many ways can the letters of the word $QUICK$ be arranged in a row if the Q and the U must remain next to each other in the order QU?

(b) How many ways can the letters of the word $QUICK$ be arranged in a row if the letters QU must remain together but may be in either the order order QU or the order UQ?

3. A group of eight people are attending the movies together.

(a) Two of the eight insist on sitting together. In how many ways can the eight be seated in a row?

(b) Two of the people do not like each other and do not want to sit side-by-side. Now how many ways can the eight be seated in a row?

4. (a) Assuming that any ten digits can be used to form a telephone number, how many seven-digit telephone numbers do not have any repeated digits?

(b) How many seven-digit telephone numbers have at least one repeated digit?

(c) What is the probability that a randomly chosen seven-digit telephone number has at least one repeated digit?

5. How many integers from 1 through 100,000 contain the digit 3 exactly once?

6. How many integers from 1 through 1,000 are multiples of 4 or multiples of 7?

7*. Suppose a public opinion polltaker reports that out of a national sample of 1,200 adults

(*i*) 675 are married;

(*ii*) 682 are from 20 to 30 years old;

(*iii*) 684 are female;

(*iv*) 195 are married and are from 20 to 30 years old;

(*v*) 467 are married females;

(*vi*) 318 are females from 20 to 30 years old;

(*vii*) 165 are married females from 20 to 30 years old.

Are the polltaker's figures consistent?

8*. How many positive integers less than 1,000 have no common factors with 1,000?