THEOREM (Some Subset Relations):
1. Inclusion of Intersection: For all sets $A$ and $B$,
   
   (a) $A \cap B \subseteq A$ and (b) $A \cap B \subseteq B$.

2. Inclusion in Union: For all sets $A$ and $B$,
   
   (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$.

3. Transitive Property of Subsets: For all sets $A, B,$ and $C$,
   
   if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof:
1(a). Suppose $A$ and $B$ are any sets and suppose $x$ is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. So, $x \in A$.

1(b). Suppose ___ and ___ are any sets and suppose ___ is any element of _____. Then ____ and ____ by definition of intersection. So, ____.

2(a). Suppose ___ and ___ are any sets and suppose ___ is any element of _____. Then ____ by definition of ________.

2(b). Suppose ___ and ___ are any sets and suppose ___ is any element of _____. Then ____ by definition of ________.

3. Suppose $A, B,$ and $C$ are any sets and suppose $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in $A$ is in $C$. To this end we note that if $x \in A$, then $x \in B$ (because $A \subseteq B$) and therefore $x \in C$ (because $B \subseteq C$). Hence $A \subseteq C$. ■
EXAMPLES:
1. Prove that for all sets \( A \) and \( B \),
\[
A - B \subseteq A.
\]

**Proof:**
Suppose ___ and ___ are any sets and suppose ___ is any element of ____. Then ____ and ____ by definition of _______. So, ____.

2. Prove that for all sets \( A, B, \) and \( C \),
\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C).
\]

**Proof:**
Suppose ___, ___, and ___ are any sets.
(I). We first prove that \( A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \). Let \( x \in A \cap (B \cup C) \). By definition of intersection, \( x \in ___ \) and \( x \in ___ \). Thus \( x \in A \) and by definition of union, \( x \in B \) or ____.

Case 1 \((x \in A \text{ and } x \in B)\): In this case, by definition of intersection \( x \in ____ \), and so by definition of union, \( x \in (A \cap B) \cup (A \cap C) \).
Case 2 \((x \in A \text{ and } x \in C)\): In this case, by definition of intersection \( x \in ____ \), and so by definition of union, _____________.

Hence in either case, \( x \in (A \cap B) \cup (A \cap C) \).
(II). We now prove that \( (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \). Let \( x \in (A \cap B) \cup (A \cap C) \). By definition of union, \( x \in ____ \) or \( x \in ____ \).

Case 1 \((x \in A \cap B)\): In this case, by definition of intersection ____ and _____. Since \( x \in B \), by definition of union, \( x \in B \cup C \). Hence \( x \in A \) and \( x \in B \cup C \), and so by definition of intersection, \( x \in ____ \).
Case 2 \((x \in A \cap C)\): In this case, by definition of intersection ____ and _____. Since _____, by definition of union, __________. Hence ____ and __________, and so by definition of intersection, \( x \in ____ \).

In either case \( x \in A \cap (B \cup C) \). ■
THEOREM (Set Identities):
Let all sets referred to below be subsets of a universal set $U$.
1. Commutative Laws: For all sets $A$ and $B$,
   
   (a) $A \cap B = B \cap A$ and (b) $A \cup B = B \cup A$.

2. Associative Laws: For all sets $A$, $B$, and $C$,
   
   (a) $(A \cap B) \cap C = A \cap (B \cap C)$ and (b) $(A \cup B) \cup C = A \cup (B \cup C)$.

3. Distributive Laws: For all sets $A$, $B$, and $C$,
   
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Intersection with $U$: For all sets $A$,
   
   $A \cap U = A$.

5. Double Complement Law: For all sets $A$,
   
   $(A^c)^c = A$.

6. Idempotent Laws: For all sets $A$,
   
   (a) $A \cap A = A$ and (b) $A \cup A = A$.

7. De Morgan’s Laws: For all sets $A$ and $B$,
   
   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.

8. Union with $U$: For all sets $A$,
   
   $A \cup U = U$.

9. Absorption Laws: For all sets $A$ and $B$,
   
   (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.

10. Alternate Representation for Set Difference: For all sets $A$ and $B$,

    $A - B = A \cap B^c$. 
**EXERCISE SET:**
For all sets $A$, $B$, and $C$,

1. $(A - B) \cup (A \cap B) = A$.
2. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
3. $(A \cup B) - (C - A) = A \cup (B - C)$.
4. $(A - B) - (B - C) = A - B$.
5. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.
6. $[(A^c \cup B^c) - A]^c = A$. 
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   (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$.

3. Transitive Property of Subsets: For all sets $A, B,$ and $C$,
   
   if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

**Proof:**
1(a). Suppose $A$ and $B$ are any sets and suppose $x$ is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. So, $x \in A$.

1(b). Suppose $A$ and $B$ are any sets and suppose $x$ is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. So, $x \in B$.

2(a). Suppose $A$ and $B$ are any sets and suppose $x$ is any element of $A$. Then $x \in A \cup B$ by definition of union.

2(b). Suppose $A$ and $B$ are any sets and suppose $x$ is any element of $B$. Then $x \in A \cup B$ by definition of union.

3. Suppose $A, B,$ and $C$ are any sets and suppose $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in $A$ is in $C$. To this end we note that if $x \in A$, then $x \in B$ (because $A \subseteq B$) and therefore $x \in C$ (because $B \subseteq C$). Hence $A \subseteq C$. 