<u>THEOREM</u> (Some Subset Relations):
1. Inclusion of Intersection: For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. Inclusion in Union: For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. Transitive Property of Subsets: For all sets A, B, and C,

$$\text{if} \quad A \subseteq B \quad \text{and} \quad B \subseteq C, \quad \text{then} \quad A \subseteq C.$$

<u>Proof:</u>
1(a). Suppose A and B are any sets and suppose x is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. So, $x \in A$.

1(b). Suppose __ and __ are any sets and suppose __ is any element of _____. Then _____ and _____ by definition of intersection. So, _____.

2(a). Suppose __ and __ are any sets and suppose __ is any element of _____. Then _____ by definition of __________.

2(b). Suppose __ and __ are any sets and suppose __ is any element of _____. Then _____ by definition of __________.

3. Suppose A, B, and C are any sets and suppose $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in A is in C. To this end we note that if $x \in A$, then $x \in B$ (because $A \subseteq B$) and therefore $x \in C$ (because $B \subseteq C$). Hence $A \subseteq C$. ■

EXAMPLES:

1. Prove that for all sets A and B,

$$A - B \subseteq A.$$

Proof:
Suppose __ and __ are any sets and suppose __ is any element of _____. Then _____ and _____ by definition of _________. So, _____.

2. Prove that for all sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof:
Suppose __, __, and __ are any sets.
(I). We first prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. By definition of intersection, $x \in$ _____ and $x \in$ _____. Thus $x \in A$ and by definition of union, $x \in B$ or _____.
Case 1 ($x \in A$ and $x \in B$): In this case, by definition of intersection $x \in$ _____, and so by definition of union, $x \in (A \cap B) \cup (A \cap C)$.
Case 2 ($x \in A$ and $x \in C$): In this case, by definition of intersection $x \in$ _____, and so by definition of union, __________________.
Hence in either case, $x \in (A \cap B) \cup (A \cap C)$.
(II). We now prove that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. By definition of union, $x \in$ _____ or $x \in$ _____.
Case 1 ($x \in A \cap B$): In this case, by definition of intersection _____ and _____ . Since $x \in B$, by definition of union, $x \in B \cup C$. Hence $x \in A$ and $x \in B \cup C$, and so by definition of intersection, $x \in$ _________.
Case 2 ($x \in A \cap C$): In this case, by definition of intersection _____ and _____ . Since _____, by definition of union, _________. Hence _____ and _________, and so by definition of intersection, $x \in$ _________.
In either case $x \in A \cap (B \cup C)$. ■

THEOREM (Set Identities):
Let all sets referred to below be subsets of a universal set U.
1. Commutative Laws: For all sets A and B,

$$\text{(a) } A \cap B = B \cap A \quad \text{and} \quad \text{(b) } A \cup B = B \cup A.$$

2. Associative Laws: For all sets A, B, and C,

$$\text{(a) } (A \cap B) \cap C = A \cap (B \cap C) \quad \text{and} \quad \text{(b) } (A \cup B) \cup C = A \cup (B \cup C).$$

3. Distributive Laws: For all sets A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and} \quad \text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. Intersection with U: For all sets A,

$$A \cap U = A.$$

5. Double Complement Law: For all sets A,

$$(A^c)^c = A.$$

6. Idempotent Laws: For all sets A,

$$\text{(a) } A \cap A = A \quad \text{and} \quad \text{(b) } A \cup A = A.$$

7. De Morgan's Laws: For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

8. Union with U: For all sets A,

$$A \cup U = U.$$

9. Absorption Laws: For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

10. Alternate Representation for Set Difference: For all sets A and B,

$$A - B = A \cap B^c.$$

EXERCISE SET:

For all sets A, B, and C,

1. $(A - B) \cup (A \cap B) = A$.

2. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

3. $(A \cup B) - (C - A) = A \cup (B - C)$.

4. $(A - B) - (B - C) = A - B$.

5. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

6. $[(A^c \cup B^c) - A]^c = A$.

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$$\text{if} \quad A \subseteq B \quad \text{and} \quad B \subseteq C, \quad \text{then} \quad A \subseteq C.$$

<u>Proof:</u>
1(a). Suppose A and B are any sets and suppose x is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. So, $x \in A$.

1(b). Suppose A and B are any sets and suppose x is any element of $A \cap B$. Then $x \in A$ and $x \in B$ by definition of intersection. So, $x \in B$.

2(a). Suppose A and B are any sets and suppose x is any element of A. Then $x \in A \cup B$ by definition of union.

2(b). Suppose A and B are any sets and suppose x is any element of B. Then $x \in A \cup B$ by definition of union.

3. Suppose A, B, and C are any sets and suppose $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in A is in C. To this end we note that if $x \in A$, then $x \in B$ (because $A \subseteq B$) and therefore $x \in C$ (because $B \subseteq C$). Hence $A \subseteq C$.