I. RATIONAL NUMBERS

1. $\forall \boldsymbol{\alpha} \in \mathbb{Z},\ \boldsymbol{\alpha} \in \mathbb{Q}$. (TRUE)

Proof. We have
$$\alpha = \frac{\alpha}{1}.$$
Therefore, since $\alpha \in \mathbb{Z},\ 1 \in \mathbb{Z}$, and $1 \neq 0$, by the definition of rational numbers we get $\alpha \in \mathbb{Q}$. ■

2. $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q},\ \boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathbb{Q}$. (TRUE)

Proof. We have
$$\alpha = \frac{p_1}{q_1}, \quad \beta = \frac{p_2}{q_2},$$
hence
$$\alpha + \beta = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}.$$
Since $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, it follows that $p_1 q_2 + p_2 q_1 \in \mathbb{Z}$ and $q_1 q_2 \in \mathbb{Z}$. Plus, since $\alpha, \beta \in \mathbb{Q}$, it follows that $q_1 \neq 0$ and $q_2 \neq 0$, therefore $q_1 q_2 \neq 0$. By this and the definition of rational numbers we get $\alpha + \beta \in \mathbb{Q}$. ■

3. $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q},\ \boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{Q}$. (TRUE)

Proof. We have
$$\alpha = \frac{p_1}{q_1}, \quad \beta = \frac{p_2}{q_2},$$
hence
$$\alpha - \beta = \frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}.$$
Since $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, it follows that $p_1 q_2 - p_2 q_1 \in \mathbb{Z}$ and $q_1 q_2 \in \mathbb{Z}$. Plus, since $\alpha, \beta \in \mathbb{Q}$, it follows that $q_1 \neq 0$ and $q_2 \neq 0$, therefore $q_1 q_2 \neq 0$. By this and the definition of rational numbers we get $\alpha - \beta \in \mathbb{Q}$. ■

4. $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q},\ \boldsymbol{\alpha} \cdot \boldsymbol{\beta} \in \mathbb{Q}$. (TRUE)

Proof. We have
$$\alpha = \frac{p_1}{q_1}, \quad \beta = \frac{p_2}{q_2},$$
hence
$$\alpha \cdot \beta = \frac{p_1 p_2}{q_1 q_2}.$$
Since $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, it follows that $p_1 p_2 \in \mathbb{Z}$ and $q_1 q_2 \in \mathbb{Z}$. Plus, since $\alpha, \beta \in \mathbb{Q}$, it follows that $q_1 \neq 0$ and $q_2 \neq 0$, therefore $q_1 q_2 \neq 0$. By this and the definition of rational numbers we get $\alpha \cdot \beta \in \mathbb{Q}$. ■

5. $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q},\ \boldsymbol{\alpha}/\boldsymbol{\beta} \in \mathbb{Q}$**. (FALSE)**

Counter-Example. Put

$$\alpha = 1, \quad \beta = 0.$$

This gives the counter-example, for we can't divide by zero.

6. $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q}$**, if** $\boldsymbol{\beta} \neq \mathbf{0}$ **then** $\boldsymbol{\alpha}/\boldsymbol{\beta} \in \mathbb{Q}$**. (TRUE)**

Proof. We have

$$\alpha = \frac{p_1}{q_1}, \quad \beta = \frac{p_2}{q_2},$$

hence

$$\alpha/\beta = \frac{p_1 q_2}{q_1 p_2}.$$

Since $p_1, p_2, q_1, q_2 \in \mathbb{Z}$, it follows that $p_1 q_2 \in \mathbb{Z}$ and $q_1 p_2 \in \mathbb{Z}$. Plus, since $\alpha \in \mathbb{Q}$ and $\beta \neq 0$, it follows that $q_1 \neq 0$ and $p_2 \neq 0$, therefore $q_1 p_2 \neq 0$. By this and the definition of rational numbers we get $\alpha/\beta \in \mathbb{Q}$. ■

7. $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Q},\ \boldsymbol{\alpha}^{\boldsymbol{\beta}} \in \mathbb{Q}$**. (FALSE)**

Counter-Example. Put

$$\alpha = 2, \quad \beta = 1/2,$$

hence

$$\alpha^\beta = 2^{1/2} = \sqrt{2}.$$

On the one hand, 2 and 1/2 are both rational numbers. On the other hand, $\sqrt{2}$ is irrational (see below). This is a contradiction.

8. $\forall \boldsymbol{\alpha} \in \mathbb{Q}$**, if** $\boldsymbol{\beta} \in \mathbb{Z}^+$ **then** $\boldsymbol{\alpha}^{\boldsymbol{\beta}} \in \mathbb{Q}$**. (TRUE)**

Proof. We have

$$\alpha = \frac{p_1}{q_1}, \quad \beta \in \mathbb{Z}^+,$$

hence

$$\alpha^\beta = \underbrace{\frac{p_1}{q_1} \cdots \frac{p_1}{q_1}}_{\beta \text{ times}} = \frac{p_1^\beta}{q_1^\beta}.$$

Since $p_1, q_1 \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^+$, it follows that $p_1^\beta \in \mathbb{Z}$ and $q_1^\beta \in \mathbb{Z}$. Plus, since $\alpha \in \mathbb{Q}$, it follows that $q_1 \neq 0$, therefore $q_1^\beta \neq 0$. By this and the definition of rational numbers we get $\alpha^\beta \in \mathbb{Q}$. ■

II. INEQUALITIES

1. $\forall \boldsymbol{a, b, c, d} \in \mathbb{R}^+$**, if** $\boldsymbol{\frac{a}{b} < \frac{c}{d}}$ **then:**

(a) $\boldsymbol{\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}}$**; (TRUE)**

Proof 1. We have

$$\frac{a}{b} < \frac{c}{d}. \tag{$*$}$$

Multiplying both sides of $(*)$ by bd (note, that bd is positive), we get

$$ad < bc. \tag{$**$}$$

Adding ab to both sides of $(**)$, we get

$$ab + ad < ab + bc,$$

which we rewrite by factoring a and b as

$$a(b+d) < b(a+c).$$

If we divide both sides of this inequality by $b(b+d)$ (note, that $b(b+d)$ is positive), we get

$$\frac{a}{b} < \frac{a+c}{b+d}. \tag{$***$}$$

Similarly, adding cd to both sides of $(**)$, we get

$$cd + ad < cd + bc,$$

which we rewrite by factoring d and c as

$$d(a+c) < c(b+d).$$

If we divide both sides of this inequality by $d(b+d)$ (note, that $d(b+d)$ is positive), we get

$$\frac{a+c}{b+d} < \frac{c}{d}. \tag{$****$}$$

It is clear, that $(***)$ and $(****)$ give the desired result.■

Proof 2 (Short Version Of Proof 1). We have

$$\frac{a}{b} < \frac{c}{d} \xrightarrow[bd>0]{\text{multiply by } bd} ad < bc \xrightarrow{+ab} ab + ad < ab + bc \xrightarrow[\text{factor } b]{\text{factor } a} a(b+d) < b(a+c).$$

If we divide both sides of the last inequality by $b(b+d)$ (note, that $b(b+d)$ is positive), we get

$$\frac{a}{b} < \frac{a+c}{b+d}. \tag{$*$}$$

Similarly,

$$\frac{a}{b} < \frac{c}{d} \xrightarrow[bd>0]{\text{multiply by } bd} ad < bc \xrightarrow{+cd} ad + cd < bc + cd \xrightarrow[\text{factor } d]{\text{factor } c} d(a+c) < c(b+d).$$

If we divide both sides of the last inequality by $d(b+d)$ (note, that $d(b+d)$ is positive), we get

$$\frac{a+c}{b+d} < \frac{c}{d}. \qquad (**)$$

It is clear, that $(*)$ and $(**)$ give the desired result.■

Proof 3 (Correct Students Version Of Proof 1). We have

$$(*) \qquad \frac{a}{b} < \frac{a+c}{b+d} \xleftarrow[b(b+d)>0]{\text{divide by } b(b+d)} a(b+d) < b(a+c) \xleftarrow[\text{factor } b]{\text{factor } a} ab+ad < ab+bc \xleftarrow{+ab} ad < bc.$$

The last inequality follows from

$$\frac{a}{b} < \frac{c}{d}.$$

Similarly,

$$(**) \qquad \frac{a+c}{b+d} < \frac{c}{d} \xleftarrow[d(b+d)>0]{\text{divide by } d(b+d)} d(a+c) < c(b+d) \xleftarrow[\text{factor } d]{\text{factor } c} ad+cd < bc+cd \xleftarrow{+cd} ad < bc.$$

The last inequality follows from

$$\frac{a}{b} < \frac{c}{d}.$$

It is clear, that $(*)$ and $(**)$ give the desired result.■

Proof 4 (Incorrect Students Version Of Proof 1). We have

$$\frac{a}{b} < \frac{a+c}{b+d} \xrightarrow[b(b+d)>0]{\text{multiply by } b(b+d)} a(b+d) < b(a+c) \xrightarrow{\text{expand}} ab+ad < ab+bc \xrightarrow{-ab} ad < bc...$$

(b) $\boldsymbol{\frac{a}{b} < \frac{at+c}{bt+d} < \frac{c}{d}}$, where $\boldsymbol{t > 0}$; (TRUE)

Proof (C.S.V.) We have

$$(*) \qquad \frac{a}{b} < \frac{at+c}{bt+d} \xleftarrow[b(bt+d)>0]{\text{divide by } b(bt+d)} a(bt+d) < b(at+c) \xleftarrow[\text{factor } b]{\text{factor } a} abt+ad < abt+bc \xleftarrow{+abt} ad < bc.$$

The last inequality follows from

$$\frac{a}{b} < \frac{c}{d}.$$

Similarly,

$$(**) \qquad \frac{at+c}{bt+d} < \frac{c}{d} \xleftarrow[d(bt+d)>0]{\text{divide by } d(bt+d)} d(at+c) < c(bt+d) \xleftarrow[\text{factor } d]{\text{factor } c} adt+cd < bct+cd \xleftarrow{+cd} adt < bct.$$

The last inequality follows from

$$\frac{a}{b} < \frac{c}{d}.$$

It is clear, that $(*)$ and $(**)$ give the desired result.■

(c) $\boldsymbol{\frac{a}{b} < \frac{at+c\gamma}{bt+d\gamma} < \frac{c}{d}}$**, where** $\boldsymbol{t, \gamma > 0}$**; (TRUE)**

Proof (C.S.V.) We have

(∗)
$$\frac{a}{b} < \frac{at+c\gamma}{bt+d\gamma} \xleftarrow[b(bt+d\gamma)>0]{\text{divide by } b(bt+d\gamma)} a(bt+d\gamma) < b(at+c\gamma) \xleftarrow[\text{factor } b]{\text{factor } a} abt+ad\gamma < abt+bc\gamma \xleftarrow{+abt} ad\gamma < bc\gamma.$$

The last inequality follows from
$$\frac{a}{b} < \frac{c}{d}.$$

Similarly,

(∗∗)
$$\frac{at+c\gamma}{bt+d\gamma} < \frac{c}{d} \xleftarrow[d(bt+d\gamma)>0]{\text{divide by } d(bt+d\gamma)} d(at+c\gamma) < c(bt+d\gamma) \xleftarrow[\text{factor } d]{\text{factor } c} adt+cd\gamma < bct+cd\gamma \xleftarrow{+cd\gamma} adt < bct.$$

The last inequality follows from
$$\frac{a}{b} < \frac{c}{d}.$$

It is clear, that (∗) and (∗∗) give the desired result.■

(d) $\boldsymbol{\frac{a}{b} < \frac{at+c\gamma}{bt+d\gamma} < \frac{c}{d}}$**, where** $\boldsymbol{t, \gamma \neq 0}$**; (FALSE)**

Counter-Example. Put

$$a = 1,\ b = 2,\ c = 1,\ d = 1,\ t = 1,\ \gamma = -1.$$

On the one hand,
$$\frac{a}{b} = \frac{1}{2} < \frac{1}{1} = \frac{c}{d}.$$

On the other hand,
$$\frac{1}{2} \not< \frac{1\cdot 1 + 1\cdot(-1)}{2\cdot 1 + 1\cdot(-1)} < \frac{1}{1}.$$

(e) $\boldsymbol{\frac{a}{b} < \frac{at_1+c\gamma_1}{bt_2+d\gamma_2} < \frac{c}{d}}$**, where** $\boldsymbol{t_1, t_2, \gamma_1, \gamma_2 > 0}$**. (FALSE)**

Counter-Example. Put

$$a = 1,\ b = 2,\ c = 1,\ d = 1,\ t_1 = t_2 = 1,\ \gamma_1 = 1,\ \gamma_1 = 3.$$

On the one hand,
$$\frac{a}{b} = \frac{1}{2} < \frac{1}{1} = \frac{c}{d}.$$

On the other hand,
$$\frac{1}{2} \not< \frac{1\cdot 1 + 1\cdot 1}{2\cdot 1 + 1\cdot 3} < \frac{1}{1}.$$

2. $\forall a, b, c \in \mathbb{R}^+$ we have

(a) $a + b \geq 2\sqrt{ab}$. (TRUE)

Proof 1 (C.S.V.) We have

$$a + b \geq 2\sqrt{ab} \xleftarrow[\text{all } >0]{\text{root}} (a+b)^2 \geq 4ab \xleftarrow{\text{factor}} a^2 + 2ab + b^2 \geq 4ab \xleftarrow{+4ab} a^2 - 2ab + b^2 \geq 0.$$

The last inequality is true, since

$$a^2 - 2ab + b^2 = (a-b)^2,$$

which is always nonnegative. ∎

Proof 2 (Indirect Proof). Suppose, contrary to our claim, that

$$\exists\ a, b \in \mathbb{R}^+ \mid a + b < 2\sqrt{ab}.$$

We have

$$a + b < 2\sqrt{ab} \xrightarrow[\text{all } >0]{\text{square}} (a+b)^2 < 4ab \xrightarrow{\text{expand}} a^2 + 2ab + b^2 < 4ab \xrightarrow{-4ab} a^2 - 2ab + b^2 < 0.$$

We obtain a contradiction, since the last inequality is false. In fact,

$$a^2 - 2ab + b^2 = (a-b)^2,$$

which is always nonnegative. ∎

(b) $a + \frac{1}{a} \geq 2$. (TRUE)

Proof (Indirect Proof). Suppose, contrary to our claim, that

$$\exists\ a, b \in \mathbb{R}^+ \mid a + \frac{1}{a} < 2.$$

We have

$$a + \frac{1}{a} < 2 \longrightarrow \frac{a^2+1}{a} < 2 \xrightarrow[a>0]{\times a} a^2 + 1 < 2a \longrightarrow a^2 - 2a + 1 < 0.$$

We obtain a contradiction, since the last inequality is false. In fact,

$$a^2 - 2a + 1 = (a-1)^2,$$

which is always nonnegative. ∎

(c) $a + \frac{1}{a} \geq 3$. (FALSE)

Counter-Example. Put $a = 1$, then

$$1 + \frac{1}{1} = 2 < 3.$$

(d) $\boldsymbol{\frac{a}{b} + \frac{b}{a} \geq 2}$**. (TRUE)**

Proof. Put $x = \frac{a}{b}$, then

$$\frac{a}{b} + \frac{b}{a} = x + \frac{1}{x},$$

which is ≥ 2 by (b). ■

(e)* $\boldsymbol{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9}$**. (TRUE)**

Proof. Expanding parentheses, we obtain

$$\begin{aligned}(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) &= \frac{a}{a} + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{b} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + \frac{c}{c} \\ &= 1 + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + 1 + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + 1 \\ &= 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right).\end{aligned}$$

Each pair is ≥ 2 by (d), therefore

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 3 + 2 + 2 + 2 = 9. \blacksquare$$

(f)* $\boldsymbol{a^2 + b^2 + c^2 \geq ab + bc + ca}$**. (TRUE)**

Proof. By (a) we have

$$\forall x, y \in \mathbb{R}^+,\ x^2 + y^2 \geq 2xy,$$

therefore

$$a^2 + b^2 \geq 2ab, \quad b^2 + c^2 \geq 2bc, \quad c^2 + a^2 \geq 2ca. \tag{$*$}$$

If we add together all the inequalities $(*)$, we obtain

$$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca,$$

and the result follows. ■

(g)* $\boldsymbol{ab(a+b) + bc(b+c) + ac(a+c) \geq 6abc}$**. (TRUE)**

Proof. We have

$$\begin{aligned}ab(a+b) + bc(b+c) + ac(a+c) &= a^2b + ab^2 + b^2c + bc^2 + a^2c + ac^2 \\ &= (a^2b + bc^2) + (ab^2 + ac^2) + (b^2c + a^2c) \\ &= b(a^2 + c^2) + a(b^2 + c^2) + c(b^2 + a^2).\end{aligned}$$

By $(*)$ the right-hand side is $\geq b2ac + a2bc + c2ba = 6abc$. ■

(h)** $\boldsymbol{a^3+b^3+c^3 \geq 3abc}$. (TRUE)

Proof. We have

$$2a^3+2b^3+2c^3$$

$$=(a^3+b^3)+(b^3+c^3)+(a^3+c^3)$$

$$=(a+b)(a^2-ab+b^2)+(b+c)(b^2-bc+c^2)+(a+c)(a^2-ac+c^2).$$

$$=(a+b)(a^2+b^2-ab)+(b+c)(b^2+c^2-bc)+(a+c)(a^2+c^2-ac).$$

By $(*)$ the right-hand side is

$$\geq (a+b)(2ab-ab)+(b+c)(2bc-bc)+(a+c)(2ac-ac)$$

$$=(a+b)ab+(b+c)bc+(a+c)ac,$$

which is $\geq 6abc$ by (g). So, we have

$$2a^3+2b^3+2c^3 \geq 6abc,$$

and the result follows. ■

(i)** $\boldsymbol{\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3}$. (TRUE)

Proof. It follows from (h) that

$$\forall\ x,y,z \in R^+,\ x+y+z \geq 3\sqrt[3]{xyz}. \qquad (*)$$

Put

$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{a}.$$

By this and $(*)$ we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}} = 3\sqrt[3]{\frac{abc}{bca}} = 3. \ ■$$

(j)** $\boldsymbol{\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}}$. (TRUE)

Proof. Put

$$x=b+c, \quad y=c+a, \quad z=a+b. \qquad (*)$$

From this it follows that

$$2a=y+z-x, \quad 2b=x+z-y, \quad 2c=x+y-z. \qquad (**)$$

By $(*)$ and $(**)$ we have

$$\begin{aligned}
&\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}\\
&=\frac{y+z-x}{x}+\frac{x+z-y}{y}+\frac{x+y-z}{z}\\
&=\frac{y}{x}+\frac{z}{x}-\frac{x}{x}+\frac{x}{y}+\frac{z}{y}-\frac{y}{y}+\frac{x}{z}+\frac{y}{z}-\frac{z}{z}\\
&=\frac{y}{x}+\frac{z}{x}-1+\frac{x}{y}+\frac{z}{y}-1+\frac{x}{z}+\frac{y}{z}-1\\
&=\left(\frac{y}{x}+\frac{x}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)+\left(\frac{z}{y}+\frac{y}{z}\right)-3,
\end{aligned}$$

which is

$$\geq 2+2+2-3=3$$

by (d). So,

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}\geq 3,$$

and the result folows. ■

III. DIVISIBILITY

$\forall a, b, c, k \in \mathbb{Z}^+$ we have:

1. $(3k+1)(3k+2)(3k+3)$ is divisible by 3.

2. If $n = 4k+1$, then 8 divides $n^2 - 1$.

3. If $a|b$ and $a|c$, then $a|(b+c)$.

4. Let $a, b \in \mathbb{Z}, a \neq 0, b \neq 0$. If $a|b$ and $b|a$, then $a = b$ or $a = -b$.

5. If $a - b \neq 0$, then $(a-b)|(a^2 - b^2)$.

6*. $(a^2 + a + 1)|(a^3 - 1)$.

7*. $(a+1)|(ab + a + b + 1)$.

8. $(a^2 + b^2 + ab)|(a^4 + a^2b^2 + b^4)$.**

9. $3 \not| k^2 - 2$.**

10. $4 \not| k^2 - 3$.**

11. $4 \not| a^2 + b^2 - 3$.**

12. $8k + 7 \neq a^2 + b^2 + c^2$.**

IV. IRRATIONALITY

1. $\sqrt{2} \notin \mathbb{Q}$.

2. $5 + \sqrt{2} \notin \mathbb{Q}$.

3*. $\sqrt{3} \notin \mathbb{Q}$.

4*. $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

5*. $\log_2 3 \notin \mathbb{Q}$.

6. $\sqrt{2} + \sqrt[3]{3} \notin \mathbb{Q}$.**

V. EXTRA INEQUALITIES

$\forall a, b, c \in \mathbb{R}^+$ we have:

1. $\dfrac{3}{a+b+c} < \dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a}$.

2*. $a^2(1+b^2) + b^2(1+c^2) + c^2(1+a^2) \geq 6abc$.

3*. $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$.

4*. If $a+b \geq 1$, **then** $a^4 + b^4 \geq 1/8$.