Mathematical Induction

I. Prove by induction the following identities:

1. $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}.$

2. $1 + 3 + 5 + \ldots + (2n-1) = n^2.$

3. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}.$

4. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1)n = \dfrac{n(n-1)(n+1)}{3}.$

5. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{(n-1)n} = \dfrac{n-1}{n}.$

6. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \ldots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}.$

7. $\left(1 - \dfrac{1}{2}\right)\left(1 - \dfrac{1}{3}\right)\left(1 - \dfrac{1}{4}\right) \ldots \left(1 - \dfrac{1}{n}\right) = \dfrac{1}{n}.$

8*. $\dfrac{1}{\sqrt{1} + \sqrt{2}} + \dfrac{1}{\sqrt{2} + \sqrt{3}} + \dfrac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \dfrac{1}{\sqrt{n-1} + \sqrt{n}} = \sqrt{n} - 1.$

9**. $\sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots + \sin n\varphi = \dfrac{\sin\dfrac{n\varphi}{2}\sin\dfrac{(n+1)\varphi}{2}}{\sin\dfrac{\varphi}{2}}.$

II. Prove by induction the following inequalities:

1. $n < 2^n$ for any integer $n \geq 1$.

2. $n^2 < n!$ for any integer $n \geq 4$.

3. $2^n < n!$ for any integer $n \geq 4$.

4. $3^n < n!$ for any integer $n \geq 7$.

5. $n^n \geq n!$ for any integer $n \geq 1$.

6. $2^{n+2} \geq 2n + 5$ for any integer $n \geq 1$.

7*. $(2n)! < 2^{2n}(n!)^2$ for any integer $n \geq 1$.

8**. $(n+1)^n < n^{n+1}$ for any integer $n \geq 3$.

9**. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2$ for any integer $n \geq 1$.

10**. $\frac{a_1^n + a_2^n}{2} \geq \left(\frac{a_1 + a_2}{2}\right)^n$ for any positive numbers a_1, a_2 and for any integer $n \geq 1$.

III. Prove by induction the following problems:

1. $n^3 - n$ is divisible by 3 for any positive integer n.

2. $n^5 - n$ is divisible by 5 for any positive integer n.

3. $3^{2n+3} + 40n - 27$ is divisible by 64 for any positive integer n.

4. $5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19 for any positive integer n.

5**. $\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ is integer for any positive integer n.