1. Exercise set 3.2

28: Suppose c is a root of a (nonzero) polynomial with rational coefficients:

$$P(c) = \frac{a_n}{b_n}c^n + \cdots + \frac{a_1}{b_1}c + \frac{a_0}{b_0} = 0,$$

where $a_i, b_i \in \mathbb{Z}$, $b_i \neq 0$, and $a_n \neq 0$. We can multiply this equation by $b_n \cdots b_1 b_0 \neq 0$, to get:

$$P'(c) = d_n c^n + \cdots + d_1 c + d_0 = 0,$$

where $d_i = b_n \cdots b_1 b_0 \frac{a_i}{b_i}$ is clearly an integer for all i, and $d_n \neq 0$. Thus, c is a root of the (nonzero) polynomial $P'(X)$, whose coefficients are integers. □

29: This is "arguing from examples": it is proving the statement for a particular case.

†32: This is "begging the question": it is assuming that $r + s$ is rational, and using it to prove that $r + s$ is rational.

2. Exercise set 3.3

4: Yes: $2m(2m+2) = 2m \cdot 2(m+1) = 4m(m+1)$, which is divisible by 4 since $m(m+1)$ is an integer.

†12: Yes: $n^2 - 1 = (4k+3)^2 - 1 = 16k^2 + 24k + 9 - 1 = 8(2k^2 + 3k + 1)$.

†15: Let a, b and c be integers, and assume $a \mid b$ and $a \mid c$. By definition, there are integers r and s such that $b = a \cdot r$, $c = a \cdot s$. Therefore, $b - c = a \cdot r - a \cdot s = a \cdot (r - s)$, with $r - s$ an integer. We conclude that $a \mid b - c$.

24: False. For instance, let $a = 6$, $b = 2$, and $c = 3$. Then $a \mid bc$, but neither $a \mid b$ nor $a \mid c$. (The statement is true if we ask for a to be a prime number.)

3. Exercise set 3.4

2: $56 = 5 \cdot 11 + 1$.

†18: Let m and $m+1$ be the two consecutive integers. We have two cases:

Case 1: m even. Then $m = 2k$ for some integer k, and $m(m+1) = 2k(2k+1)$ is even, by definition.

Case 2: m odd. Then $m = 2k+1$ for some integer k, and $m(m+1) = (2k+1)(2k+2) = 2(2k+1)(k+1)$, which is also even.

Therefore, $m(m+1)$ is even. □

20: Apply the Quotient-Remainder Theorem with $d = 3$. This implies that there exist integers q and r such that $n = 3q + r$, and $0 \leq r < 3$. But the only nonnegative integers less than 3 are 0, 1, and 2. Hence, either $n = 3q$, $n = 3q + 1$, or $n = 3q + 2$. □

†28: By the Quotient-Remainder Theorem, there exist integers 1 and r sucn that $n = 4q + 4$, with $0 \leq r < 4$. Now, $n(n^2 - 1)(n+2) = (n-1)n(n+1)(n+2)$, and its easy to see that for each possible value of r, exactly one of the factors is divisible by 4.[1] Hence, their product is always divisible by 4. □

4. Exercise set 3.6

†4: Negation: "There is a positive rational number x, which is the least positive rational number, i.e. for any positive rational number y, $x \leq y$". Suppose such an x exists. Then, for all positive rational numbers y, $x \leq y$. In particular, for $y = x/2$, we have $x \leq x/2$, which is a contradiction (since x is positive). □

21: False. For instance, $\sqrt{2}$ is irrational, and $\sqrt{2} \times \sqrt{2} = 2$, which is rational.

22: Suppose that $c = a + br$ were rational. Say $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, and $c = \frac{c_1}{c_2}$, with $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbf{Z}$. Since $b \neq 0$, we can write

$$r = \frac{c-a}{b} = \frac{\frac{c_1}{c_2} - \frac{a_1}{a_2}}{\frac{b_1}{b_2}} = \frac{\frac{c_1 a_2 - c_2 a_1}{c_2 a_2}}{\frac{b_1}{b_2}} = \frac{b_2(c_1 a_2 - c_2 a_1)}{b_1 c_2 a_2},$$

which is rational, by definition (note that $c_2 \neq 0$ and $a_2 \neq 0$ for they are denominators of rational numbers, while $b_1 \neq 0$ since $b \neq 0$.)

[1] If $r = 0$, then n is divisible by 4; if $r = 1$, then $n-1$ is divisible by 4; if $r = 2$, then $n+2$ is divisible by 4; and if $r = 3$, then $n+1$ is divisible by 4 (prove it!).