1. Exercise set 4.1

2: $b_0 = 1 + 2^0 = 2, b_1 = 1 + 2^1 = 3, b_2 = 1 + 2^2 = 5, b_3 = 1 + 2^3 = 9.$

†15: One solution is $a_i = (-1)^i \frac{i+1}{i+2}$ for $i \geq 0$.

†21: $\sum_{m=0}^{4} \frac{1}{2^m} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16}.$

†34: $\prod_{i=1}^{4} \left(1 - r^i\right).$

36: $\sum_{i=1}^{n} \frac{i}{(i+1)!}.$

55: $\frac{n!}{(n-k)!} = \frac{n(n-1)\cdots(n-k+1)\cancel{(n-k)\cdots 2\cdot 1}}{\cancel{(n-k)\cdots 2\cdot 1}} = n(n-1)\cdots(n-k+1).$

2. Excercise set 4.2

†7:

- The formula is true for $n = 1$:
  Certainly, $1 = 1(2 \cdot 1 - 1)$.
- If the formula is true for $n = k$, then it is true for $n = k + 1$:
  Suppose $1 + \cdots + (4k - 1) = k(2k - 1)$, for some integer $k \geq 1$. We must show that $1 + \cdots + (4k - 1) + (4(k + 1) - 1) = (k + 1)(2(k + 1) - 1)$.
  But by inductive hypothesis, the left hand side is

$$k(2k - 1) + (4(k + 1) - 3) = 2k^2 + 3k + 1,$$

while the right hand side is

$$(k + 1)(2(k + 1) - 1) = 2k^2 + 3k + 1. \qquad \square$$

10:

- The formula is true for $n=1$:
  Since $1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$.
- If the formula is true for $n=k$, then it is true for $n=k+1$:
  Suppose $1^3+\cdots+k^3 = \left[\frac{k(k+1)}{2}\right]^2$ for some integer $k \geq 1$. We compute

$$\begin{aligned} 1^3+\cdots+k^3+(k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\ &= (k+1)^2\left[\frac{k^2}{2^2}+(k+1)\right] \\ &= (k+1)^2 \frac{\overbrace{k^2+4k+4}^{(k+2)^2}}{2^2} \\ &= \left[\frac{(k+1)(k+2)}{2}\right]^2, \end{aligned}$$

where in the first equality we have used the inductive hypothesis. □

15:

- The formula is true for $n=2$: easy.
- If the formula is true for $n=k$, then it is true for $n=k+1$:

$$\begin{aligned} \prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) &= \left(1-\frac{1}{(k+1)^2}\right)\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right) \\ &= \left(\frac{(k+1)^2-1}{(k+1)^2}\right)\left(\frac{k+1}{2k}\right) \\ &= \frac{\cancel{k}(k+2)\cancel{(k+1)}}{\cancel{(k+1)}(k+1)2\cancel{k}} = \frac{(k+1)+1}{2(k+1)}, \end{aligned}$$

where the inductive hypothesis has been used in the second equality. □

3. Excercise set 4.3

9: Clearly, 7 divides $2^3-1=7$. We have $2^{3(k+1)}-1 = 8\cdot 2^{3k}-1 = 7\cdot 2^k + (2^{3k}-1)$, which is divisible by 7 if we assume $2^{3k}-1$ is, which we can by inductive hyphotesis. □

†10: Again, the base step is trivial (3 divides 3). For the inductive step observe that the difference between the term for $n = k+1$ and the term for $n = k$ is

$$\begin{aligned}\big((k+1)^3 - 7(k+1) + 3\big) &- \big(k^3 - 7k + 3\big) \\ &= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 - k^3 + 7k - 3 \\ &= 3k^2 + 3k - 6,\end{aligned}$$

which is divisible by 3. Therefore, if we assume that the term for $n = k$ is divisible by 3, we conclude that the term for $n = k+1$ is divisible by 3 (since the difference is). □

15: (sketch) The tricky part is to prove that the difference between two consecutive terms,

$$(k+1)\big((k+1)^2 + 5\big) - k\big(k^2 + 5\big) = \cdots = 3k^2 + 3k + 6,$$

is divisible by 6. We can work this out again by induction, or note that

$$\frac{3k^2 + 3k + 6}{6} = \frac{k(k+1)}{2} + 1,$$

which is an integer since one of k or $k+1$ has to be even. □

†17: The statement is true for $n = 0$: $2^0 = 1 < 2 = (0+2)!$. Assume it is true for $n = k$, for $k \geq 0$. We claim it is true for $n = k+1$. In fact, $2^{k+1} = 2 \cdot 2^k < 2(k+2)!$ by inductive hypothesis; also $2 < k+3$, since $k \geq 0$; therefore $2(k+2)! < (k+3)(k+2)! = (k+3)!$. □

20: (sketch) The basis step is just a computation. For the inductive step it suffices to see that the differences between the $n = k+1$ inequality and the $n = k$ inequality are greater (or equal) in the left hand side than in the right hand side (make sure you understand why this is enough).
(a) $(k+1)^3 - k^3 = 3k^2 + 3k + 1$, which is bigger than $(2(k+1)+1) - (2k+1) = 2$ (since $k \geq 2$). □
(b) $(k+1)! - k! = (k+1)k! - k! = k \cdot k!$, which is bigger than $(k+1)^2 - k^2 = 2k+1$ (since $k \geq 4$, $k \cdot k!$ is certainly bigger than $k \cdot 3 > 2k+1$). □