

COMMUTATIVE RINGS

Definition 1:

A commutative ring R is a set with two operations, addition and multiplication, such that:

- (i) R is an abelian group under addition;
- (ii) $ab = ba$ for all $a, b \in R$ (commutative law);
- (iii) $a(bc) = (ab)c$ for any $a, b, c \in R$ (associative law);
- (iv) there is an element $1 \in R$ with $1 \neq 0$ and with $1 \cdot a = a \cdot 1 = a$ for any $a \in R$;
- (v) $a(b + c) = ab + ac$ for any $a, b, c \in R$ (distributive law).

Example:

1. \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} are commutative rings.
2. $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is a commutative ring.
3. $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ is a commutative ring.
4. $\{a + b\sqrt[3]{2} : a, b \in \mathbb{Z}\}$ is not a ring. Moreover, $\{a + b\sqrt[3]{2} : a, b \in \mathbb{Q}\}$ is not a ring.
5. The set of all 2×2 matrices is a noncommutative ring.
6. \mathbb{Z}_m is a commutative ring.

Theorem 1:

Let R be a commutative ring. Then:

- (i) $0 \cdot a = 0$ for any $a \in R$.
- (ii) If $-a$ is that number which, when added to a , gives 0, then $(-1)(-a) = a$ for any $a \in R$.
- (iii) $(-1)a = -a$ for any $a \in R$.

Definition 2:

A subset S of a commutative ring R is a subring of R if:

- (i) $1 \in S$;
- (ii) if $a, b \in S$, then $a - b \in S$;
- (iii) if $a, b \in S$, then $ab \in S$.

Example:

1. \mathbb{Z} is a subring of \mathbb{Q} ; \mathbb{Q} is a subring of \mathbb{R} ; \mathbb{R} is a subring of \mathbb{C} ;
2. $\mathbb{Z}[i]$ is a subring of \mathbb{C} .

Definition 3:

A domain is a commutative ring R that satisfies the cancellation law for multiplication:

$$\text{if } ca = cb \text{ and } c \neq 0, \text{ then } a = b.$$

Example:

1. \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} are domains.
2. \mathbb{Z}_4 is not a domain, since from $[2][2] = [2][0]$ does not follow $[2] = [0]$.

Theorem 2:

A commutative ring R is a domain if and only if the product of any two nonzero elements of R is nonzero.

Corollary:

\mathbb{Z}_m is a domain if and only if m is a prime.

FIELDS

Definition 4:

A field F is a commutative ring such that for any nonzero $a \in F$ there exists $a^{-1} \in F$.

Example: \mathbb{Q} , \mathbb{R} , \mathbb{C} are fields. \mathbb{Z} is not a field.

Definition 5:

An element u in a commutative ring R is called a unit if there exists $u^{-1} \in R$.

Example: All nonzero elements of \mathbb{Q} , \mathbb{R} , and \mathbb{C} are units.

Definition 4':

A field F is a commutative ring in which every nonzero element of F is a unit.

Theorem 3:

Every field F is a domain.

Remark:

The converse of Theorem 3 is false. For example, \mathbb{Z} is a domain, but not a field.

Theorem 4:

The commutative ring \mathbb{Z}_m is a field if and only if m is a prime.

POLYNOMIALS

Definition 6:

If R is a commutative ring, then

$$R[x] = \{s_n x^n + \dots + s_1 x + s_0, s_i \in R\}$$

is called the ring of polynomials over R . If

$$f(x) = s_n x^n + \dots + s_1 x + s_0$$

is a polynomial over R , then s_n is called the leading coefficient and n is called the degree of $f(x)$, denoted by $\deg(f)$. If $s_n = 1$, then $f(x)$ is called a monic polynomial.

Theorem 5 (Division Algorithm):

Assume that k is a field and that $f(x), g(x) \in k[x]$ with $f(x) \neq 0$. Then there are unique polynomials $q(x), r(x) \in k[x]$ such that

$$g(x) = f(x)q(x) + r(x)$$

and either $r(x) = 0$ or $\deg(r) < \deg(f)$.

Example:

Let $g(x) = x^2 - 2x + 1$, $f(x) = x - 3$, then $x^2 - 2x + 1 = (x - 3)(x + 1) + 4$.

ROOTS OF POLYNOMIALS

Definition 7:

If $f(x) \in k[x]$, where k is a field, then a root of $f(x)$ in k is an element $a \in k$ with $f(a) = 0$.

Example:

The polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$ has no roots in \mathbb{R} . However, since $f(x)$ is also from $\mathbb{C}[x]$, it has roots $x_1 = i$, $x_2 = -i$ in \mathbb{C} .

Lemma 1:

Let k be a field and let $f(x) \in k[x]$. Then for any $a \in k$ there exists $q(x) \in k[x]$ such that

$$f(x) = (x - a)q(x) + f(a).$$

Proof:

By the Division Algorithm we have

$$f(x) = (x - a)q(x) + r(x),$$

where $\deg(r) < \deg(x - a) = 1$, and therefore $\deg(r) = 0$, i.e. $r(x) = r$ is a constant. So,

$$f(x) = (x - a)q(x) + r \implies f(a) = (a - a)q(a) + r = r = r(x). \blacksquare$$

Lemma 2:

Let k be a field and let $f(x) \in k[x]$. Then $a \in k$ is a root of $f(x)$ in k if and only if $x - a$ divides $f(x)$ in $k[x]$, i.e. there exists $q(x) \in k[x]$ such that

$$f(x) = (x - a)q(x).$$

Proof:

\implies) If a is a root of $f(x)$ in k , then $f(a) = 0$, therefore by Lemma 1 we get

$$f(x) = (x - a)q(x) + f(a) = (x - a)q(x).$$

\impliedby) If $f(x) = (x - a)q(x)$, then

$$f(a) = (a - a)q(a) = 0,$$

which means that a is a root of $f(x)$. ■

Theorem 6:

Let k be a field and let $f(x) \in k[x]$. Let also $f(x)$ has degree n . Then $f(x)$ has at most n roots in k .

Proof:

We will use induction by n . If $n = 0$, then $f(x)$ is a constant, and we are done. Suppose the theorem is true for some $n = m \geq 0$. We prove it for $n = m + 1$. In fact, if $f(x)$ has no roots in k , then we are done, since $0 < n$. Otherwise, we may assume that there exists $a \in k$ such that $f(a) = 0$. Then by Lemma 2 we have

$$f(x) = q(x)(x - a) \quad \text{and} \quad \deg(q) = m.$$

If there is a root $b \in k$ with $b \neq a$, then

$$0 = f(b) = q(b)(b - a).$$

Since $b - a \neq 0$, we have $q(b) = 0$, so that b is a root of $q(x)$. Now $\deg(q) = m$, so that the inductive hypothesis says that $q(x)$ has at most m roots in k . Therefore, $f(x)$ has at most $m + 1$ roots in k . ■

Corollary 1:

Let k be a field and let $f(x), g(x) \in k[x]$. If $\deg(f) \leq \deg(g) = n$ and if $f(a) = g(a)$ for $n + 1$ values $a \in k$, then $f(x) = g(x)$.

Proof:

Suppose to the contrary that $f(x) \neq g(x)$. Then

$$h(x) = f(x) - g(x) \neq 0$$

and

$$\deg(h) \leq \max(\deg(f), \deg(g)) = n.$$

By hypothesis, there are $n + 1$ elements $a \in k$ such that

$$h(a) = f(a) - g(a) = 0,$$

which contradicts Theorem 6. ■.

Corollary 2:

Let k be a field and let $f(x) \in k[x]$. Let also $f(x)$ has degree n and $\alpha_1, \dots, \alpha_n$ are distinct roots of $f(x)$ in k . Then there exists $c \in k$ such that

$$f(x) = c(x - \alpha_1) \dots (x - \alpha_n).$$

GREATEST COMMON DIVISOR

Definition 8:

Let k be a field and let $f(x), g(x) \in k[x]$. A common divisor of $f(x)$ and $g(x)$ is a polynomial $c(x) \in k[x]$ such that $c(x) \mid f(x)$ and $c(x) \mid g(x)$. The greatest common divisor (gcd) is a monic common divisor of the highest degree.

Theorem 7:

Let k be a field and let $f(x), g(x) \in k[x]$. Then their gcd is a linear combination of $f(x)$ and $g(x)$.

Proof:

Part I. Consider the following set:

$$I = \{s(x)f(x) + t(x)g(x) : s(x), t(x) \in k[x]\}.$$

Pick a polynomial $d(x) \in I$ of the smallest degree. We have

$$d(x) = s(x)f(x) + t(x)g(x). \tag{*}$$

By the Division Algorithm we obtain

$$f(x) = d(x)q(x) + r(x), \quad \text{where } r(x) = 0 \quad \text{or} \quad \deg(r) < \deg(d). \tag{**}$$

From (*) and (**) it follows that

$$\begin{aligned} r(x) &= f(x) - d(x)q(x) = f(x) - [s(x)f(x) + t(x)g(x)]q(x) \\ &= [1 - s(x)q(x)]f(x) - t(x)q(x)g(x) \in I. \end{aligned}$$

So, $r(x)$ is a linear combination of $f(x)$ and $g(x)$ and $\deg(r) < \deg(d)$. This is a contradiction. Therefore $r(x) = 0$, which means $d(x) \mid f(x)$. A similar argument shows that $d(x) \mid g(x)$.

Part II. We now prove that $d(x)$ is the greatest common divisor of $f(x)$ and $g(x)$. In fact, assume to the contrary that $c(x)$ is the gcd of $f(x)$ and $g(x)$. Then $c(x) \mid f(x)$ and $c(x) \mid g(x)$. Hence

$$c(x) \mid d(x) = s(x)f(x) + t(x)g(x).$$

Since $c(x)$ and $d(x)$ are monic polynomials, it follows that $\deg(c) < \deg(d)$, which is a contradiction. ■

EUCLIDEAN ALGORITHM

Theorem 8 (Euclidean Algorithm):

Let k be a field and let $f(x), g(x) \in k[x]$. Then there is an algorithm for computing the gcd $(f(x), g(x))$.

Proof:

To find the greatest common divisor of two polynomials $f(x)$ and $g(x)$ we apply the algorithm, similar to the Euclidean Algorithm in \mathbb{Z} :

$$\begin{aligned}g(x) &= f(x)q_1(x) + r_1(x) \\f(x) &= r_1(x)q_2(x) + r_2(x) \\r_1(x) &= r_2(x)q_3(x) + r_3(x) \\&\dots \\r_{n-2}(x) &= r_{n-1}(x)q_n(x) + r_n(x) \\r_{n-1}(x) &= r_n(x)q_{n+1}(x),\end{aligned}$$

therefore the gcd $(f(x), g(x)) = r_n(x)$.

Example:

Find the gcd $(x^3 + 1, x^5 + 1)$ and express it as a linear combination of $x^3 + 1$ and $x^5 + 1$.

Solution:

We have

$$\begin{aligned}x^5 + 1 &= (x^3 + 1)x^2 + (-x^2 + 1) \\x^3 + 1 &= (-x^2 + 1)(-x) + (x + 1) \\-x^2 + 1 &= (x + 1)(-x + 1)\end{aligned}$$

therefore the gcd $(x^3 + 1, x^5 + 1) = x + 1$. Finally, it follows that

$$x+1 = (x^3+1) - (-x^2+1)(-x) = (x^3+1) - [x^5+1 - (x^3+1)x^2](-x) = (x^3+1)(-x^3+1) + (x^5+1)x.$$

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Example:

1. Z, Q, R, C are commutative rings.
2. $Z[i] = \{a + bi : a, b \in Z\}$ is a commutative ring.
3. $\{a + b\sqrt{2} : a, b \in Z\}$ is a commutative ring.
4. $\{a + b\sqrt[3]{2} : a, b \in Z\}$ is not a ring. Moreover, $\{a + b\sqrt[3]{2} : a, b \in Q\}$ is not a ring.
5. The set of all 2×2 matrices is a noncommutative ring.
6. Z_m is a commutative ring.

Theorem 1:

Let R be a commutative ring. Then:

(i) $0 \cdot a = 0$ for any $a \in R$.

(ii) If $-a$ is that number which, when added to a , gives 0, then $(-1)(-a) = a$ for any $a \in R$.

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Definition 2:

A subset S of a commutative ring R is a subring of R if:

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- (ii) if $a, b \in S$, then $a - b \in S$;
- (iii) if $a, b \in S$, then $ab \in S$.

Example:

1. Z is a subring of Q ; Q is a subring of R ; R is a subring of C ;
2. $Z[i]$ is a subring of C .

Definition 3:

A domain is a commutative ring R that satisfies the cancellation law for multiplication:

if $ca = cb$ and $c \neq 0$, then $a = b$.

Example:

1. Z , Q , R , C are domains.
2. Z_4 is not a domain, since from

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does not follow $[2] = [0]$.

Theorem 2:

A commutative ring R is a domain if and only if the product of any two nonzero elements of R is nonzero.

Corollary:

Z_m is a domain if and only if m is a prime.

Definition 4:

A field F is a commutative ring such that for any nonzero $a \in F$ there exists $a^{-1} \in F$.

Example:

Q, R, C are fields. Z is not a field.

Definition 5:

An element u in a commutative ring R is called a unit if there exists $u^{-1} \in R$.

Example:

All nonzero elements of Q , R , and C are units.

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Remark:

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Definition:

If R is a commutative ring, then

$$R[x] = \{s_n x^n + \dots + s_1 x + s_0, s_i \in R\}$$

is called the ring of polynomials over R .

If

$$f(x) = s_n x^n + \dots + s_1 x + s_0$$

is a polynomial over R , then s_n is called the leading coefficient and n is called the degree of $f(x)$, denoted by $\deg(f)$.

If $s_n = 1$, then $f(x)$ is called a monic polynomial.

Theorem (Division Algorithm):

Assume that k is a field and that

$$f(x), g(x) \in k[x]$$

with $f(x) \neq 0$. Then there are unique polynomials $q(x), r(x) \in k[x]$ with

$$g(x) = f(x)q(x) + r(x)$$

and either $r(x) = 0$ or $\deg(r) < \deg(f)$.

Example:

Let $g(x) = x^2 - 2x + 1$, $f(x) = x - 3$,
then

$$x^2 - 2x + 1 = (x - 3)(x + 1) + 4.$$

Definition:

If $f(x) \in k[x]$, where k is a field, then a root of $f(x)$ in k is an element $a \in k$ with $f(a) = 0$.

Example:

The polynomial $f(x) = x^2 + 1 \in R[x]$ has no roots in R . However, since $f(x)$ is also from $C[x]$, it has roots $x_1 = i$, $x_2 = -i$ in C .

Lemma 1:

Let k be a field and let $f(x) \in k[x]$. Then for any $a \in k$ there exists $q(x) \in k[x]$ such that

$$f(x) = (x - a)q(x) + f(a).$$

Proof:

By the Division Algorithm we have

$$f(x) = (x - a)q(x) + r(x),$$

where $\deg(r) < \deg(x - a) = 1$, and therefore $\deg(r) = 0$, i.e. $r(x) = r$ is a constant. So,

$$f(x) = (x - a)q(x) + r \quad \implies \quad f(a) = (a - a)q(a) + r$$

Lemma 2:

Let k be a field and let $f(x) \in k[x]$. Then $a \in k$ is a root of $f(x)$ in k if and only if $x - a$ divides $f(x)$ in $k[x]$, i.e. there exists $q(x) \in k[x]$ such that

$$f(x) = (x - a)q(x).$$

Proof:

\implies) If a is a root of $f(x)$ in k , then $f(a) = 0$, therefore by Lemma 1 we get $f(x) = (x - a)q(x) + f(a) = (x - a)q(x)$.

\impliedby) If $f(x) = (x - a)q(x)$, then

$$f(a) = (a - a)q(a) = 0,$$

which means that a is a root of $f(x)$. ■

Theorem 6:

Let k be a field and let $f(x) \in k[x]$. Let also $f(x)$ has degree n . Then $f(x)$ has at most n roots in k .

Proof:

We will use induction by n . If $n = 0$, then $f(x)$ is a constant, and we are done. Suppose the theorem is true for some $n = m \geq 0$. We prove it for $n = m + 1$. In fact, if $f(x)$ has no roots in k , then we are done, since $0 < n$. Otherwise, we may assume that there exists $a \in k$ such that $f(a) = 0$. Then by Lemma 2 we have

$$f(x) = q(x)(x - a) \quad \text{and} \quad \deg(q) = m.$$

If there is a root $b \in k$ with $b \neq a$, then

$$0 = f(b) = q(b)(b - a).$$

Since $b - a \neq 0$, we have $q(b) = 0$, so that b is a root of $q(x)$. Now $\deg(q) =$

m , so that the inductive hypothesis says that $q(x)$ has at most m roots in k . Therefore, $f(x)$ has at most $m + 1$ roots in k . ■

Corollary 1:

Let k be a field and let $f(x), g(x) \in k[x]$. If $\deg(f) \leq \deg(g) = n$ and if $f(a) = g(a)$ for $n + 1$ values $a \in k$, then $f(x) = g(x)$.

Proof:

Suppose to the contrary that $f(x) \neq g(x)$. Then

$$h(x) = f(x) - g(x) \neq 0$$

and

$$\deg(h) \leq \max(\deg(f), \deg(g)) = n.$$

By hypothesis, there are $n + 1$ elements $a \in k$ such that

$$h(a) = f(a) - g(a) = 0,$$

which contradicts Theorem 6. ■.

Corollary 2:

Let k be a field and let $f(x) \in k[x]$. Let also $f(x)$ has degree n and $\alpha_1, \dots, \alpha_n$ are distinct roots of $f(x)$ in k . Then there exists $c \in k$ such that

$$f(x) = c(x - \alpha_1) \dots (x - \alpha_n).$$

GREATEST COMMON DIVISOR

Definition 8:

Let k be a field and let $f(x), g(x) \in k[x]$. A common divisor of $f(x)$ and $g(x)$ is a polynomial $c(x) \in k[x]$ such that $c(x) \mid f(x)$ and $c(x) \mid g(x)$. The greatest common divisor (gcd) is a monic common divisor of the highest degree.

Theorem 7:

Let k be a field and let $f(x), g(x) \in k[x]$. Then their gcd is a linear combination

of $f(x)$ and $g(x)$.

Proof:

Part I. Consider the following set:

$$I = \{s(x)f(x) + t(x)g(x) : s(x), t(x) \in k[x]\}.$$

Pick a polynomial $d(x) \in I$ of the smallest degree. We have

$$d(x) = s(x)f(x) + t(x)g(x). \quad (*)$$

By the Division Algorithm we obtain

$$f(x) = d(x)q(x) + r(x), \quad \text{where } r(x) = 0 \quad (**)$$

From (*) and (**) it follows that

$$\begin{aligned} r(x) &= f(x) - d(x)q(x) = f(x) - [s(x)f(x) + t(x)g(x)]q(x) \\ &= [1 - s(x)q(x)]f(x) - t(x)g(x)q(x) \end{aligned}$$

So, $r(x)$ is a linear combination of $f(x)$ and $g(x)$ and $\deg(r) < \deg(d)$. This is a contradiction. Therefore $r(x) = 0$, which means $d(x) \mid f(x)$. A similar argument shows that $d(x) \mid g(x)$.

Part II. We now prove that $d(x)$ is the greatest common divisor of $f(x)$ and $g(x)$.

In fact, assume to the contrary that $c(x)$ is the gcd of $f(x)$ and $g(x)$. Then $c(x) \mid f(x)$ and $c(x) \mid g(x)$. Hence

$$c(x) \mid d(x) = s(x)f(x) + t(x)g(x).$$

Since $c(x)$ and $d(x)$ are monic polynomials, it follows that $\deg(c) < \deg(d)$, which is a contradiction. ■

EUCLIDEAN ALGORITHM

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Let k be a field and let $f(x), g(x) \in k[x]$. Then there is an algorithm for computing the gcd $(f(x), g(x))$.

Proof:

To find the greatest common divisor of two polynomials $f(x)$ and $g(x)$ we apply the algorithm, similar to the Euclidean

Algorithm in \mathbb{Z} :

$$\begin{aligned}g(x) &= f(x)q_1(x) + r_1(x) \\f(x) &= r_1(x)q_2(x) + r_2(x) \\r_1(x) &= r_2(x)q_3(x) + r_3(x)\end{aligned}$$

...

$$\begin{aligned}r_{n-2}(x) &= r_{n-1}(x)q_n(x) + r_n(x) \\r_{n-1}(x) &= r_n(x)q_{n+1}(x),\end{aligned}$$

therefore the gcd $(f(x), g(x)) = r_n(x)$.

Example:

Find the gcd $(x^3 + 1, x^5 + 1)$ and express it as a linear combination of $x^3 + 1$ and $x^5 + 1$.

Solution:

We have

$$\begin{aligned}x^5 + 1 &= (x^3 + 1)x^2 + (-x^2 + 1) \\x^3 + 1 &= (-x^2 + 1)(-x) + (x + 1) \\-x^2 + 1 &= (x + 1)(-x + 1)\end{aligned}$$

therefore the gcd $(x^3 + 1, x^5 + 1) = x + 1$.

Finally, it follows that

$$x+1 = (x^3+1) - (-x^2+1)(-x) = (x^3+1) - [x^5$$