Permutations

<u>**Definition**</u>**:**
A **permutation** of a set X is a rearrangement of its elements.

<u>**Example**</u>**:**
Let $X = \{1, 2, 3\}$. Then there are 6 permutations:

$$123,\ 132,\ 213,\ 231,\ 312,\ 321.$$

<u>**Definition′**</u>**:**
A **permutation** of a set X is a one-one correspondence (a bijection) from X to itself.

<u>**Notation**</u>**:**
Let $X = \{1, 2, \ldots, n\}$ and $\alpha : X \to X$ be a permutation. It is convenient to describe this function in the following way:

$$\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha(1) & \alpha(2) & \ldots & \alpha(n) \end{pmatrix}.$$

<u>**Example**</u>**:**

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

<u>**Definition**</u>**:**
Let $X = \{1, 2, \ldots, n\}$ and $\alpha : X \to X$ be a permutation. Let $i_1, i_2, \ldots, i_r$ be distinct numbers from $\{1, 2, \ldots, n\}$. If

$$\alpha(i_1) = i_2, \quad \alpha(i_2) = i_3,\ \ldots,\ \alpha(i_{r-1}) = i_r, \quad \alpha(i_r) = i_1,$$

and $\alpha(i_\nu) = i_\nu$ for other numbers from $\{1, 2, \ldots, n\}$, then α is called an ***r*-cycle**.

<u>**Notation**</u>**:**
An r-cycle is denoted by $(i_1\ i_2 \ldots i_r)$.

<u>**Example**</u>**:**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix} = (125) \quad 3 - \text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} \quad \text{is not a cycle}$$

Remark:

We can use different notations for the same cycles. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1) = (2) = (3), \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) = (231) = (312).$$

Warning:

Do not confuse notations of a permutation and a cycle. For example,

$$(123) \neq 123.$$

Instead, $(123) = 231$ and $123 = (1)$.

Composition (Product) Of Permutations

Let

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n) \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & \dots & n \\ \beta(1) & \beta(2) & \dots & \beta(n) \end{pmatrix}.$$

Then

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha(\beta(1)) & \alpha(\beta(2)) & \dots & \alpha(\beta(n)) \end{pmatrix},$$

$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ \beta(\alpha(1)) & \beta(\alpha(2)) & \dots & \beta(\alpha(n)) \end{pmatrix}.$$

Example:

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$. We have:

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix},$$

$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}.$$

Remark:

It is convenient to represent a permutation as the product of circles.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 6 & 9 & 5 & 7 & 1 & 8 & 4 \end{pmatrix} = (1367)(49)(2)(5)(8) = (1367)(49)$$

Remark:
One can find a composition of permutations using circles.

Example:
1. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} = (1532)(4) = (1532),$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} = (14)(2)(35) = (14)(35).$$

We have:

$$\alpha \circ \beta = (1532)(14)(35) = (1452)(3) = (1452) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix},$$

$$\beta \circ \alpha = (14)(35)(1532) = (1324)(5) = (1324) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}.$$

2. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 6 & 1 & 7 & 3 & 2 \end{pmatrix} = (15724)(36)$. Find α^{-1}. We have:

$$\alpha^{-1} = (42751)(63)$$

In fact,

$$\alpha \circ \alpha^{-1} = (15724)(36)(42751)(63) = (1)$$

and

$$\alpha^{-1} \circ \alpha = (42751)(63)(15724)(36) = (1).$$

Theorem:
The inverse of the cycle $\alpha = (i_1 i_2 \dots i_r)$ is the cycle $\alpha^{-1} = (i_r i_{r-1} \dots i_1)$.

Theorem:
Every permutation α is either a cycle or a product of disjoint (with no common elements) cycles.

Solutions

1. Determine which permutations are equal:

(a) $(12) \neq 12$

(b) $(1) = 12$

(c) $(1)(2) = (1)$

(d) $(12)(34) \neq (1234)$

(e) $(12)(34) = (123)(234)$

(f) $(12)(34) \neq (123)(234)(341)$

(g) $(124)(53) = (53)(124)$

(h) $(124)(53) = (124)(35)$

(i) $(124)(53) \neq (142)(53)$

(j) $(12345) \neq 12345$

(k) $(12345) = 23451$

(l) $(23451) = 23451$

2. Factor the following permutations into the product of cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 5 & 4 & 6 & 7 & 8 \end{pmatrix} = (4\ 5)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 3 & 10 & 4 & 11 & 12 & 6 & 9 & 1 & 2 & 8 & 7 \end{pmatrix} = (1\ 5\ 11\ 8\ 9)(2\ 3\ 10)(6\ 12\ 7)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 2 & 12 & 7 & 9 & 14 & 8 & 4 & 5 & 3 & 6 & 10 & 11 & 13 & 15 \end{pmatrix} = (3\ 12\ 10)(4\ 7\ 8)(5\ 9)(6\ 14\ 13\ 11)$$

3. Find the following products:

$$(12)(34)(56)(1234) = (24)(56)$$
$$(12)(23)(34)(45) = (12345)$$
$$(12)(34)(56) = (12)(34)(56)$$
$$(123)(234)(345) = (12)(45)$$

4. Let $\alpha = (135)(24)$, $\beta = (124)(35)$. Find:
   (a) $\alpha\beta = (143)$
   (b) $\beta\alpha = (152)$
   (c) $\beta^{-1} = (421)(53)$
   (d) $\alpha^{2004} = (1)$

Problems

1. Factor the following permutations into the product of cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & 3 & 6 & 15 & 8 & 2 & 7 & 11 & 10 & 9 & 1 & 12 & 13 & 14 & 4 \end{pmatrix}$$

2. Find the following products:

$$(134)(23)(235)(45)$$
$$(24)(35)(12345)$$
$$(345)(234)(125)$$

3. Let $\alpha = (124)(3567)$. Find:
   (a) α^{-1}
   (b) α^{12}

Groups

Definition:

An operation on a set G is a function $* : G \times G \to G$.

Definition:

A group is a set G which is equipped with an operation $*$ and a special element $e \in G$, called the identity, such that

(i) the associative law holds: for every $x, y, z \in G$,

$$x * (y * z) = (x * y) * z;$$

(ii) $e * x = x = x * e$ for all $x \in G$;

(iii) for every $x \in G$, there is $x' \in G$ with $x * x' = e = x' * x$.

Example:

Set	Operation "+"	Operation "*"	Additional Condition

Example:

Set	Operation "+"	Operation "*"

Example:

Set	Operation "+"	Operation "*"
$\{2n : n \in \mathbb{Z}\}$		
$\{2n+1 : n \in \mathbb{Z}\}$		
$\{3n : n \in \mathbb{Z}\}$		
$\{kn : n \in \mathbb{Z}\}$, where $k \in \mathbb{N}$ is some fixed number		
$\{a^n : n \in \mathbb{Z}\}$, where $a \in \mathbb{R}, a \neq 0, \pm 1$, is some fixed number		
$\left\{\frac{p}{2^n} : p \in \mathbb{N}, n \in \mathbb{Z}_{\geq 0}\right\}$		

Example:

Set	Operation
$\mathbb{R}_{>0}$	$a * b = a^b$
$\mathbb{R}_{>0}$	$a * b = a^2 b^2$

Definition:

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$$x * (y * z) = (x * y) * z;$$

(ii) $e * x = x = x * e$ for all $x \in G$;

(iii) for every $x \in G$, there is $x' \in G$ with $x * x' = e = x' * x$.

Set	"+"	"*"	Add. Cond.
$\boldsymbol{N}$			
$\boldsymbol{Z}$			
$\boldsymbol{Q}$			
$\boldsymbol{R}$			
$\boldsymbol{R} \setminus \boldsymbol{Q}$			

Set	“+”	“*”
$\boldsymbol{Z}_{>0}$		
$\boldsymbol{Z}_{\geq 0}$		
$\boldsymbol{Q}_{>0}$		
$\boldsymbol{Q}_{\geq 0}$		
$\boldsymbol{R}_{>0}$		
$\boldsymbol{R}_{\geq 0}$		

Set	"+"	"*"
$\{2n : n \in Z\}$		
$\{2n+1 : n \in Z\}$		
$\{3n : n \in Z\}$		
$\{kn : n \in Z\}$		
$\{a^n : n \in Z\}$, where $a \neq 0, \pm 1$		
$\left\{\frac{p}{2^n} : p \in N, n \in Z_{\geq 0}\right\}$		

Set	**Operation**
$\boldsymbol{R}_{>0}$	$\boldsymbol{a * b = a^b}$
$\boldsymbol{R}_{>0}$	$\boldsymbol{a * b = a^2 b^2}$