Permutations

Definition:
A **permutation** of a set X is a rearrangement of its elements.

Example:

1. Let $X = \{1, 2\}$. Then there are 2 permutations:
$$12,\ 21.$$
2. Let $X = \{1, 2, 3\}$. Then there are 6 permutations:
$$123,\ 132,\ 213,\ 231,\ 312,\ 321.$$
3. Let $X = \{1, 2, 3, 4\}$. Then there are 24 permutations:
$$\begin{array}{cccccc} 1234, & 1243, & 1324, & 1342, & 1423, & 1432 \\ 2134, & 2143, & 2314, & 2341, & 2413, & 2431 \\ 3214, & 3241, & 3124, & 3142, & 3421, & 3412 \\ 4231, & 4213, & 4321, & 4312, & 4123, & 4132 \end{array}$$

Remark:
One can show that there are exactly $n!$ permutations of the n-element set X.

Definition′:
A **permutation** of a set X is a one-one correspondence (a bijection) from X to itself.

Notation:
Let $X = \{1, 2, \ldots, n\}$ and $\alpha : X \to X$ be a permutation. It is convenient to describe this function in the following way:
$$\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ \alpha(1) & \alpha(2) & \ldots & \alpha(n) \end{pmatrix}.$$

Example:
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$$

Conclusion:
For a permutation we can use two different notations. For example, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$ and 24513 are the same permutations.

Definition:

Let $X = \{1, 2, \ldots, n\}$ and $\alpha : X \to X$ be a permutation. Let $i_1, i_2, \ldots, i_r$ be distinct numbers from $\{1, 2, \ldots, n\}$. If

$$\alpha(i_1) = i_2, \quad \alpha(i_2) = i_3, \; \ldots, \; \alpha(i_{r-1}) = i_r, \quad \alpha(i_r) = i_1,$$

and $\alpha(i_\nu) = i_\nu$ for other numbers from $\{1, 2, \ldots, n\}$, then α is called an ***r*-cycle**.

Notation:

An r-cycle is denoted by $(i_1 \; i_2 \ldots i_r)$.

Example:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1) \quad 1-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = (1) \quad 1-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (12) \quad 2-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13) \quad 2-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) \quad 3-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = (1423) \quad 4-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix} = (13425) \quad 5-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix} = (125) \quad 3-\text{cycle}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} \quad \text{is not a cycle}$$

Remark:

We can use different notations for the same cycles. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1) = (2) = (3), \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) = (231) = (312).$$

Warning:

Do not confuse notations of a permutation and a cycle. For example,

$$(123) \neq 123.$$

Instead, $(123) = 231$ and $123 = (1)$.

Composition (Product) Of Permutations

Let
$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha(1) & \alpha(2) & \dots & \alpha(n) \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & \dots & n \\ \beta(1) & \beta(2) & \dots & \beta(n) \end{pmatrix}.$$
Then
$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha(\beta(1)) & \alpha(\beta(2)) & \dots & \alpha(\beta(n)) \end{pmatrix},$$
$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & \dots & n \\ \beta(\alpha(1)) & \beta(\alpha(2)) & \dots & \beta(\alpha(n)) \end{pmatrix}.$$

Warning:
In general, $\alpha \circ \beta \neq \beta \circ \alpha$.

Example:

1. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. We have:
$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$
$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

2. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$. We have:
$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix},$$
$$\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}.$$

3. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 6 & 1 & 7 & 3 & 2 \end{pmatrix}$. Find α^{-1}. We have:
$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & 2 & 1 & 3 & 5 \end{pmatrix}.$$

In fact,
$$\alpha \circ \alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 6 & 1 & 7 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & 2 & 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix},$$
$$\alpha^{-1} \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 6 & 2 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 6 & 1 & 7 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Remark:
It is convenient to represent a permutation as the product of circles.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 6 & 9 & 5 & 7 & 1 & 8 & 4 \end{pmatrix} = (1367)(49)(2)(5)(8) = (1367)(49)$$

Remark:
One can find a composition of permutations using circles.

Example:

1. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3) = (12)$. We have:

$$\alpha \circ \beta = (123)(12) = (13)(2) = (13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$\beta \circ \alpha = (12)(123) = (1)(23) = (23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

2. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 4 & 3 \end{pmatrix} = (1532)(4) = (1532),$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} = (14)(2)(35) = (14)(35).$$

We have:

$$\alpha \circ \beta = (1532)(14)(35) = (1452)(3) = (1452) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix},$$

$$\beta \circ \alpha = (14)(35)(1532) = (1324)(5) = (1324) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}.$$

3. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 6 & 1 & 7 & 3 & 2 \end{pmatrix} = (15724)(36)$. Find α^{-1}. We have:

$$\alpha^{-1} = (42751)(63)$$

In fact,

$$\alpha \circ \alpha^{-1} = (15724)(36)(42751)(63) = (1)$$

and

$$\alpha^{-1} \circ \alpha = (42751)(63)(15724)(36) = (1).$$

Theorem:
The inverse of the cycle $\alpha = (i_1 i_2 \ldots i_r)$ is the cycle $\alpha^{-1} = (i_r i_{r-1} \ldots i_1)$.

Problems

1. Determine which permutations are equal:

   (a) (12) and 12

   (b) (1) and 12

   (c) (1)(2) and (1)

   (d) (12)(34) and (1234)

   (e) (12)(34) and (123)(234)

   (f) (12)(34) and (123)(234)(341)

   (g) (124)(53) and (53)(124)

   (h) (124)(53) and (124)(35)

   (i) (124)(53) and (142)(53)

   (j) (12345) and 12345

   (k) (12345) and 23451

   (l) (23451) and 23451

2. Factor the following permutations into the product of cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 5 & 4 & 6 & 7 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 3 & 10 & 4 & 11 & 12 & 6 & 9 & 1 & 2 & 8 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 2 & 12 & 7 & 9 & 14 & 8 & 4 & 5 & 3 & 6 & 10 & 11 & 13 & 15 \end{pmatrix}$$

3. Find the following products:

   (12)(34)(56)(1234)
   (12)(23)(34)(45)
   (12)(34)(56)
   (123)(234)(345)

4. Let $\alpha = (135)(24)$, $\beta = (124)(35)$. Find:

   (a) $\alpha\beta$

   (b) $\beta\alpha$

   (c) β^{-1}

   (d) α^{2004}