<u>Theorem 1</u> : We have

$$3 \mid n^3 - n \qquad (*)$$

for any integer $n \geq 1$.

<u>Proof</u> :

STEP1 : For $n = 1$ $(*)$ is true, since

$$3 \mid 1^3 - 1.$$

STEP2 : Suppose $(*)$ is true for some $n = k \geq 1$, that is

$$3 \mid k^3 - k.$$

STEP3 : Prove that $(*)$ is true for $n = k + 1$:

$$3 \mid (k+1)^3 - (k+1).$$

We have :

$$\begin{aligned} &(k+1)^3 - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= \underbrace{k^3 - k}_{\substack{\text{St. 2} \\ \text{div. by 3}}} + \underbrace{3k^2 + 3k}_{\text{div. by 3}}. \ \blacksquare \end{aligned}$$

Theorem 2 : We have

$$5 \mid n^5 - n \qquad (*)$$

for any integer $n \geq 1$.

Proof :

STEP1 : For $n = 1$ $(*)$ is true, since

$$5 \mid 1^5 - 1.$$

STEP2 : Suppose $(*)$ is true for some $n = k \geq 1$, that is

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STEP3 : Prove that $(*)$ is true for $n = k + 1$:

$$5 \mid (k+1)^5 - (k+1).$$

We have :

$$\begin{aligned} &(k+1)^5 - (k+1) \\ &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= \underbrace{k^5 - k}_{\substack{\text{St. 2} \\ \text{div. by 5}}} + \underbrace{5k^4 + 10k^3 + 10k^2 + 5k}_{\text{div. by 5}}. \ \blacksquare \end{aligned}$$

Theorem 3 : We have

$$7 \mid n^7 - n \qquad (*)$$

for any integer $n \geq 1$.

Proof :

STEP1 : For $n = 1$ $(*)$ is true, since

$$7 \mid 1^7 - 1.$$

STEP2 : Suppose $(*)$ is true for some $n = k \geq 1$, that is

$$7 \mid k^7 - k.$$

STEP3 : Prove that $(*)$ is true for $n = k + 1$:

$$7 \mid (k+1)^7 - (k+1).$$

We have :

$$(k+1)^7 - (k+1)$$
$$= k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1$$
$$= \underbrace{k^7 - k}_{\substack{\text{St. 2} \\ \text{div. by 7}}} + \underbrace{7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k}_{\text{div. by 7}}. \blacksquare$$

Theorem 4 : Let p be a prime. We have

$$p \mid n^p - n \qquad (*)$$

for any integer $n \geq 1$.

Proof :

STEP1 : For $n = 1$ $(*)$ is true, since

$$p \mid 1^p - 1.$$

STEP2 : Suppose $(*)$ is true for some $n = k \geq 1$, that is

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We have :

$$(k+1)^p - (k+1)$$

$$= k^p + \binom{p}{1} k^{p-1} + \ldots + \binom{p}{p-1} k + 1 - k - 1$$

$$= \underbrace{k^p - k}_{\substack{\text{St. 2} \\ \text{div. by p}}} + \underbrace{\binom{p}{1} k^{p-1} + \ldots + \binom{p}{p-1} k}_{\text{div. by p}} . \blacksquare$$

Fermat's Little Theorem

Theorem 1: We have

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Theorem 2: We have

$$5 \mid n^5 - n \tag{$*$}$$

for any integer $n \geq 1$.

Proof:

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We have

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= \underbrace{k^5 - k}_{\substack{\text{St. 2} \\ \text{div. by 5}}} + \underbrace{5k^4 + 10k^3 + 10k^2 + 5k}_{\text{div. by 5}}. \blacksquare\end{aligned}$$

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