TWO PROBLEMS

<u>PROBLEM 1</u>: Prove that $\sqrt{2}$ is irrational.

<u>Proof</u>: Assume to the contrary that $\sqrt{2}$ is rational, that is

$$\sqrt{2} = \frac{a}{b},$$

where a and b are integers and $b \neq 0$. Moreover, let <u>a and b have no common divisor > 1</u>. Then

$$2 = \frac{a^2}{b^2} \quad \Rightarrow \quad 2b^2 = a^2. \tag{1}$$

Since $2b^2$ is even, it follows that a^2 is even. Then <u>a is also even</u> (in fact, if a is odd, then a^2 is odd). This means that there exists $q \in \mathbb{Z}$ such that

$$a = 2q. \tag{2}$$

Substituting (2) into (1), we get

$$2b^2 = (2q)^2 \quad \Rightarrow \quad 2b^2 = 4q^2 \quad \Rightarrow \quad b^2 = 2q^2.$$

Since $2q^2$ is even, it follows that b^2 is even. Then <u>b is also even</u> by the arguments above. This is a contradiction. ■

<u>PROBLEM 2</u>: Prove that $\sqrt{3}$ is irrational.

<u>Proof</u>: Assume to the contrary that $\sqrt{3}$ is rational, that is

$$\sqrt{3} = \frac{a}{b},$$

where a and b are integers and $b \neq 0$. Moreover, let <u>a and b have no common divisor > 1</u>. Then

$$3 = \frac{a^2}{b^2} \quad \Rightarrow \quad 3b^2 = a^2. \tag{1}$$

Since $3b^2$ is ...???

Proof (continuation): ... Since $3b^2$ is divisible by 3, it follows that a^2 is divisible by 3. Then a is also divisible by 3 (in fact, if a is not divisible by 3, then ...???)

DIVISION ALGORITHM

DEFINITION:

If a and b are integers with $a \neq 0$, we say that $\underline{a \text{ divides } b}$ if there exists an integer q such that $b = aq$.

THEOREM (DIVISION ALGORITHM): *For any integers a and b with $a \neq 0$ there exist unique integers q and r such that*

$$b = aq + r, \quad \textit{where} \quad 0 \leq r < |a|.$$

The integers q and r are called the **quotient** *and the* **reminder** *respectively.*

EXAMPLE 1: Let $b = 49$ and $a = 4$, then $49 = 4 \cdot 12 + 1$, so the quotient is 12 and the reminder is 1.

REMARK: We can also write 49 as $3 \cdot 12 + 13$, but in this case 13 is not a reminder, since it is NOT less than 3.

EXAMPLE 2: Let $a = 2$. Since $0 \leq r < 2$, then for any integer number b we have ONLY TWO possibilities:

$$b = 2q \quad \text{or} \quad b = 2q + 1.$$

So, thanks to the Division Algorithm we proved that any integer number is either even or odd.

EXAMPLE 3: Let $a = 3$. Since $0 \leq r < 3$, then for any integer number b we have ONLY THREE possibilities:

$$b = 3q, \quad b = 3q + 1, \quad \text{or} \quad b = 3q + 2.$$

Proof of Problem 2: Assume to the contrary that $\sqrt{3}$ is rational, that is

$$\sqrt{3} = \frac{a}{b},$$

where a and b are integers and $b \neq 0$. Moreover, let $\underline{a \text{ and } b \text{ have no common divisor} > 1}$. Then

$$3 = \frac{a^2}{b^2} \quad \Rightarrow \quad 3b^2 = a^2. \tag{1}$$

Since $3b^2$ is divisible by 3, it follows that a^2 is divisible by 3. Then $\underline{a \text{ is also divisible by } 3}$.

In fact, if a is not divisible by 3, then by the Division Algorithm (see also Example 3) there exists $q \in \mathbb{Z}$ such that...

Proof of Problem 2: Assume to the contrary that $\sqrt{3}$ is rational, that is

$$\sqrt{3} = \frac{a}{b},$$

where a and b are integers and $b \neq 0$. Moreover, let $\underline{a \text{ and } b \text{ have no common divisor} > 1}$. Then

$$3 = \frac{a^2}{b^2} \quad \Rightarrow \quad 3b^2 = a^2. \tag{1}$$

Since $3b^2$ is divisible by 3, it follows that a^2 is divisible by 3. Then $\underline{a \text{ is also divisible by } 3.}$

In fact, if a is not divisible by 3, then by the Division Algorithm there exists $q \in \mathbb{Z}$ such that

$$a = 3q + 1 \quad \text{or} \quad a = 3q + 2.$$

Suppose $a = 3q + 1$, then

$$a^2 = (3q+1)^2 = 9q^2 + 6q + 1 = 3(\underbrace{3q^2 + 2q}_{q'}) + 1 = 3q' + 1,$$

which is not divisible by 3. We get a contradiction. Similarly, if $a = 3q + 2$, then

$$a^2 = (3q+2)^2 = 9q^2 + 12q + 4 = 3(\underbrace{3q^2 + 4q + 1}_{q''}) + 1 = 3q'' + 1,$$

which is not divisible by 3. We get a contradiction again.

So, we proved that if a^2 is divisible by 3, then $\underline{a \text{ is also divisible by } 3.}$ This means that there exists $q \in \mathbb{Z}$ such that

$$a = 3q. \tag{2}$$

Substituting (2) into (1), we get

$$3b^2 = (3q)^2 \quad \Rightarrow \quad 3b^2 = 9q^2 \quad \Rightarrow \quad b^2 = 3q^2.$$

Since $3q^2$ is divisible by 3, it follows that b^2 is divisible by 3. Then $\underline{b \text{ is also divisible by } 3}$ by the arguments above. This is a contradiction. ■

PROBLEM 3: Prove that $\sqrt{101}$ is irrational.

Proof: Assume to the contrary that $\sqrt{101}$ is rational, that is

$$\sqrt{101} = \frac{a}{b},$$

where a and b are integers and $b \neq 0$. Moreover, let $\underline{a \text{ and } b \text{ have no common divisor} > 1}$. Then

$$101 = \frac{a^2}{b^2} \quad \Rightarrow \quad 101b^2 = a^2. \tag{1}$$

Since $101b^2$ is divisible by 101, it follows that a^2 is divisible by 101. Then $\underline{a \text{ is also divisible by } 101.}$

In fact, if a is not divisible by 101, then by the Division Algorithm there exists $q \in \mathbb{Z}$ such that

$$a = 101q + 1, \quad \text{or} \quad a = 101q + 2, \quad \ldots, \quad \text{or} \quad a = 101q + 100.$$

Suppose ...???

QUESTION: We should prove that if a^2 is divisible by 101, then a is also divisible by 101. Is it possible to prove it without using the division algorithm?