RATIONAL AND IRRATIONAL NUMBERS

DEFINITION:

Rational numbers are all numbers of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

EXAMPLE: $\frac{1}{2},\ -\frac{5}{3},\ 2,\ 0,\ \frac{50}{10}$, etc.

NOTATIONS:

$\mathbb{N}$ = all natural numbers, that is, $1, 2, 3, \ldots$

$\mathbb{Z}$ = all integer numbers, that is, $0, \pm 1, \pm 2, \pm 3, \ldots$

$\mathbb{Q}$ = all rational numbers

$\mathbb{R}$ = all real numbers

DEFINITION:

A number which is not rational is said to be irrational.

EXAMPLE: We prove that $\sqrt{2}$ is irrational. Assume to the contrary that $\sqrt{2}$ is rational, that is

$$\sqrt{2} = \frac{p}{q},$$

where p and q are integers and $q \neq 0$. Moreover, let p and q have no common divisor > 1. Then

$$2 = \frac{p^2}{q^2} \quad \Rightarrow \quad 2q^2 = p^2. \tag{1}$$

Since $2q^2$ is even, it follows that p^2 is even. Then p is also even (in fact, if p is odd, then p^2 is odd). This means that there exists $k \in \mathbb{Z}$ such that

$$p = 2k. \tag{2}$$

Substituting (2) into (1), we get

$$2q^2 = (2k)^2 \quad \Rightarrow \quad 2q^2 = 4k^2 \quad \Rightarrow \quad q^2 = 2k^2.$$

Since $2k^2$ is even, it follows that q^2 is even. Then q is also even. This is a contradiction. ■

EXERCISE SET: Prove that the following numbers are irrational:

1. $\sqrt[3]{4}$

2. $\sqrt{6}$

3. $\frac{1}{3}\sqrt{2} + 5$

4*. $\log_5 2$

5*. $\sqrt{2} + \sqrt{3}$

6**. $\sqrt{2} + \sqrt[3]{3}$

7***. $\sin 1^\circ$

8***. $e = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$

SOLUTIONS

1. We prove that $\sqrt[3]{4}$ is irrational. Assume to the contrary that $\sqrt[3]{4}$ is rational, that is

$$\sqrt[3]{4} = \frac{p}{q},$$

where p and q are integers and $q \neq 0$. Moreover, let $\underline{p \text{ and } q \text{ have no common divisor} > 1}$. Then

$$4 = \frac{p^3}{q^3} \quad \Rightarrow \quad 4q^3 = p^3. \tag{1}$$

Since $4q^3$ is even, it follows that p^3 is even. Then $\underline{p \text{ is also even}}$ (in fact, if p is odd, then p^3 is odd). This means that there exists $k \in \mathbb{Z}$ such that

$$p = 2k. \tag{2}$$

Substituting (2) into (1), we get

$$4q^3 = (2k)^3 \quad \Rightarrow \quad 4q^3 = 8k^3 \quad \Rightarrow \quad q^3 = 2k^3.$$

Since $2k^3$ is even, it follows that q^3 is even. Then $\underline{q \text{ is also even}}$. This is a contradiction. ■

2. We prove that $\sqrt{6}$ is irrational. Assume to the contrary that $\sqrt{6}$ is rational, that is

$$\sqrt{6} = \frac{p}{q},$$

where p and q are integers and $q \neq 0$. Moreover, let $\underline{p \text{ and } q \text{ have no common divisor} > 1}$. Then

$$6 = \frac{p^2}{q^2} \quad \Rightarrow \quad 6q^2 = p^2. \tag{1}$$

Since $6q^2$ is even, it follows that p^2 is even. Then $\underline{p \text{ is also even}}$ (in fact, if p is odd, then p^2 is odd). This means that there exists $k \in \mathbb{Z}$ such that

$$p = 2k. \tag{2}$$

Substituting (2) into (1), we get

$$6q^2 = (2k)^2 \quad \Rightarrow \quad 6q^2 = 4k^2 \quad \Rightarrow \quad 3q^2 = 2k^2.$$

Since $2k^2$ is even, it follows that $3q^2$ is even. Then $\underline{q \text{ is also even}}$ (in fact, if q is odd, then $3q^2$ is odd). This is a contradiction. ■

3. We prove that $\frac{1}{3}\sqrt{2} + 5$ is irrational. Assume to the contrary that $\frac{1}{3}\sqrt{2} + 5$ is rational, that is

$$\frac{1}{3}\sqrt{2} + 5 = \frac{p}{q},$$

where p and q are integers and $q \neq 0$. Then

$$\sqrt{2} = \frac{3(p - 5q)}{q}.$$

Since $\sqrt{2}$ is irrational and $\frac{3(p-5q)}{q}$ is rational, we obtain a contradiction. ■

4*. We prove that $\log_5 2$ is irrational. Assume to the contrary that $\log_5 2$ is rational, that is

$$\log_5 2 = \frac{p}{q},$$

where p and q are integers and $q \neq 0$. Then

$$5^{p/q} = 2 \quad \Rightarrow \quad 5^p = 2^q.$$

Since 5^p is odd and 2^q is even, we obtain a contradiction. ■

5*. We prove that $\sqrt{2}+\sqrt{3}$ is irrational. Assume to the contrary that $\sqrt{2}+\sqrt{3}$ is rational, that is

$$\sqrt{2}+\sqrt{3}=\frac{p}{q},$$

where p and q are integers and $q \neq 0$. Then

$$\left(\sqrt{2}+\sqrt{3}\right)^2=\frac{p^2}{q^2} \quad \Rightarrow \quad 2+2\sqrt{2}\sqrt{3}+3=\frac{p^2}{q^2} \quad \Rightarrow \quad 5+2\sqrt{6}=\frac{p^2}{q^2} \quad \Rightarrow \quad \sqrt{6}=\frac{p^2-5q^2}{2q^2}.$$

Since $\sqrt{6}$ is irrational and $\dfrac{p^2-5q^2}{2q^2}$ is rational, we obtain a contradiction. ■

6.** We prove that $\sqrt{2}+\sqrt[3]{3}$ is irrational. Assume to the contrary that $\sqrt{2}+\sqrt[3]{3}$ is rational, that is

$$\sqrt{2}+\sqrt[3]{3}=\frac{p}{q},$$

where p and q are integers and $q \neq 0$. It follows that

$$\sqrt[3]{3}=\frac{p}{q}-\sqrt{2},$$

hence

$$\begin{aligned}3=\left(\frac{p}{q}-\sqrt{2}\right)^3=\frac{p^3}{q^3}-3\frac{p^2}{q^2}\sqrt{2}+3\frac{p}{q}\left(\sqrt{2}\right)^2-\left(\sqrt{2}\right)^3 &= \frac{p^3}{q^3}-3\frac{p^2}{q^2}\sqrt{2}+6\frac{p}{q}-2\sqrt{2}\\ &= \frac{p^3}{q^3}+6\frac{p}{q}-\sqrt{2}\left(3\frac{p^2}{q^2}+2\right).\end{aligned}$$

We can rewrite this as

$$\sqrt{2}=\frac{\dfrac{p^3}{q^3}+6\dfrac{p}{q}-3}{3\dfrac{p^2}{q^2}+2}=\frac{p^3+6pq^2-3q^3}{3p^2q+2q^3}.$$

Since $\sqrt{2}$ is irrational and $\dfrac{p^3+6pq^2-3q^3}{3p^2q+2q^3}$ is rational, we obtain a contradiction. ■

7*.** We prove that $\sin 1^\circ$ is irrational. Assume to the contrary that $\sin 1^\circ$ is rational. Then $\cos^2 1^\circ$ and $\cos 2^\circ$ are also rational, since

$$\cos^2 1^\circ = 1-\sin^2 1^\circ \quad \text{and} \quad \cos 2^\circ=\cos^2 1^\circ-\sin^2 1^\circ.$$

Similarly, $\cos 4^\circ$, $\cos 8^\circ$, $\cos 16^\circ$, and $\cos 32^\circ$ are rational, since

$$\cos 4^\circ = 2\cos^2 2^\circ-1, \quad \cos 8^\circ=2\cos^2 4^\circ-1, \quad \cos 16^\circ = 2\cos^2 8^\circ -1, \quad \text{and} \quad \cos 32^\circ=2\cos^2 16^\circ-1.$$

On the other hand, we have

$$\begin{aligned}\frac{\sqrt{3}}{2}=\cos 30^\circ=\cos(32^\circ-2^\circ) &= \cos 32^\circ\cos 2^\circ+\sin 32^\circ \sin 2^\circ\\ &= \cos 32^\circ\cos 2^\circ+2\cos 16^\circ\sin 16^\circ\sin 2^\circ\\ &= \cos 32^\circ\cos 2^\circ+4\cos 16^\circ\cos 8^\circ\sin 8^\circ\sin 2^\circ\\ &= \cos 32^\circ\cos 2^\circ+8\cos 16^\circ\cos 8^\circ\cos 4^\circ\sin 4^\circ\sin 2^\circ\\ &= \cos 32^\circ\cos 2^\circ+16\cos 16^\circ\cos 8^\circ\cos 4^\circ\cos 2^\circ\sin^2 2^\circ\\ &= \cos 32^\circ\cos 2^\circ+64\cos 16^\circ\cos 8^\circ\cos 4^\circ\cos 2^\circ\cos^2 1^\circ\sin^2 1^\circ.\end{aligned}$$

The right-hand side is rational. One can prove that $\dfrac{\sqrt{3}}{2}$ is irrational. We obtain a contradiction. ■

8*.** We prove that $2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$ is irrational. Assume to the contrary that this number is rational, that is

$$\frac{p}{q}=2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots,$$

where p and q are integers and $q\neq 0$. We multiply both sides by $qn!$ with

$$n>q. \tag{1}$$

We get

$$\begin{aligned}
pn! &= qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots\right)\\
&= qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)+qn!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\ldots\right)\\
&= qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)+q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right),
\end{aligned}$$

so

$$pn!-qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)=q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right).$$

Note that $pn!$ and $qn!\left(2+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)$ are integer. If we prove that

$$q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right)<1,$$

we obtain a contradiction. To this end we observe that

$$\frac{1}{(n+1)(n+2)}<\frac{1}{(n+1)^2},\quad \frac{1}{(n+1)(n+2)(n+3)}<\frac{1}{(n+1)^3},\ldots.$$

By this and a formula of geometric progression we have

$$\begin{aligned}
q\left(\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right) &< q\left(\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots\right)\\
&= q\frac{1}{(n+1)\left(1-\frac{1}{n+1}\right)}\\
&= q\frac{1}{n+1-\frac{n+1}{n+1}}\\
&= q\frac{1}{n+1-1}\\
&= \frac{q}{n},
\end{aligned}$$

which is <1 by (1). ■