Mathematical Induction

Theorem 1: Prove that

$$1+2+3+\ldots+n=\frac{n(n+1)}{2} \qquad (*)$$

for any integer $n \geq 1$.
Proof:
STEP 1: For n=1 $(*)$ is true, since

$$1=\frac{1(1+1)}{2}.$$

STEP 2: Suppose $(*)$ is true for some $n=k \geq 1$, that is

$$1+2+3+\ldots+k=\frac{k(k+1)}{2}.$$

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is

$$1+2+3+\ldots+k+(k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2}.$$

We have

$$1+2+3+\ldots+k+(k+1) \stackrel{\textbf{ST.2}}{=} \frac{k(k+1)}{2}+(k+1)=(k+1)\left(\frac{k}{2}+1\right)=\frac{(k+1)(k+2)}{2}. \blacksquare$$

Theorem 2: Prove that

$$1+3+5+\ldots+(2n-1)=n^2 \qquad (*)$$

for any integer $n \geq 1$.
Proof:
STEP 1: For n=1 $(*)$ is true, since $1=1^2$.
STEP 2: Suppose $(*)$ is true for some $n=k \geq 1$, that is

$$1+3+5+\ldots+(2k-1)=k^2.$$

STEP 3: Prove that $(*)$ is true for $n=k+1$, that is

$$1+3+5+\ldots+(2k-1)+(2k+1) \stackrel{?}{=} (k+1)^2.$$

We have: $1+3+5+\ldots+(2k-1)+(2k+1) \stackrel{\textbf{ST.2}}{=} k^2+(2k+1)=(k+1)^2.$ $\blacksquare$

Theorem 3: Prove that

$$n! \le n^n \qquad (*)$$

for any integer $n \ge 1$.

Proof:

STEP 1: For n=1 $(*)$ is true, since $1! = 1^1$.

STEP 2: Suppose $(*)$ is true for some $n = k \ge 1$, that is $k! \le k^k$.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $(k+1)! \overset{?}{\le} (k+1)^{k+1}$. We have

$$(k+1)! = k! \cdot (k+1) \overset{\textbf{ST.2}}{\le} k^k \cdot (k+1) < (k+1)^k \cdot (k+1) = (k+1)^{k+1}. \blacksquare$$

Theorem 4: Prove that

$$3^{2n} - 1 \text{ div. by } 8 \qquad (*)$$

for any integer $n \ge 0$.

Proof:

STEP 1: For n=0 $(*)$ is true, since $3^0 - 1$ is divisible by 8.

STEP 2: Suppose $(*)$ is true for some $n = k \ge 0$, that is $3^{2k} - 1$ is divisible by 8.

STEP 3: Prove that $(*)$ is true for $n = k+1$, that is $3^{2(k+1)} - 1$ is divisible by 8. We have

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \cdot 9 - 1 = 3^{2k}(8+1) - 1 = \underbrace{3^{2k} \cdot 8}_{\text{div. by } 8} + \underbrace{3^{2k} - 1}_{\substack{\text{St. 2} \\ \text{div. by } 8}}. \blacksquare$$

PROBLEMS

I. Prove by induction the following identities:

1. $$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

2. $$1 + 3 + 5 + \ldots + (2n-1) = n^2.$$

3. $$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

4. $$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1)n = \frac{n(n-1)(n+1)}{3}.$$

5. $$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(n-1)n} = \frac{n-1}{n}.$$

6. $$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

7. $$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \ldots \left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

8*. $$\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{n-1} + \sqrt{n}} = \sqrt{n} - 1.$$

9**. $$\sin\varphi + \sin 2\varphi + \sin 3\varphi + \ldots + \sin n\varphi = \frac{\sin\frac{n\varphi}{2}\sin\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}.$$

II. Prove by induction the following inequalities:

1. $2^n > n$ for any integer $n \geq 1$.

2. $n! > n^2$ for any integer $n \geq 4$.

3. $2^n < n!$ for any integer $n \geq 4$.

4. $3^n < n!$ for any integer $n \geq 7$.

5. $3^n \geq 2n + 1$ for any integer $n \geq 1$.

6. $n! \leq n^n$ for any integer $n \geq 1$.

7. $2^{n+2} \geq 2n + 5$ for any integer $n \geq 1$.

8*. $(2n)! < 2^{2n}(n!)^2$ for any integer $n \geq 1$.

9**. $(n+1)^n < n^{n+1}$ for any integer $n \geq 3$.

10**. $\dfrac{a_1^n + a_2^n}{2} \geq \left(\dfrac{a_1 + a_2}{2}\right)^n$ for any positive numbers a_1, a_2 and for any integer $n \geq 1$.

11***. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2} < 2$ for any integer $n \geq 1$.

III. Prove by induction the following problems:

1. $n^3 - n$ is divisible by 3 for any nonnegative integer n.

2. $n^5 - n$ is divisible by 5 for any nonnegative integer n.

3. $n^3 - 7n + 3$ is divisible by 3 for any nonnegative integer n.

4. $4^n - 1$ is divisible by 3 for any nonnegative integer n.

5. $3^{2n} - 1$ is divisible by 8 for any positive integer n.

6. $7^n - 2^n$ is divisible by 5 for any nonnegative integer n.

7*. $3^{2n+3} + 40n - 27$ is divisible by 64 for any nonnegative integer n.

8*. $5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19 for any nonnegative integer n.

9**. $\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$ is integer for any nonnegative integer n.

Theorem : **Prove that**

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \qquad (*)$$

for any integer $n \geq 1$.

Proof :

STEP1 : For $n = 1$ $(*)$ **is true, since**

$$1 = \frac{1(1+1)}{2}.$$

STEP2 : Suppose $(*)$ **is true for some** $n = k \geq 1$, **that is**

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}.$$

STEP3 : Prove that $(*)$ **is true for** $n = k + 1$:

$$1 + 2 + 3 + \ldots + k + (k+1) \overset{?}{=} \frac{(k+1)(k+2)}{2}.$$

We have :

$$1 + 2 + 3 + \ldots + k + (k+1)$$

$$\overset{\text{ST.2}}{=} \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{(k+1)(k+2)}{2}. \; \blacksquare$$

Theorem : Prove that

$$1+3+5+\ldots+(2n-1)=n^2 \quad (*)$$

for any integer $n \geq 1$.

Proof :

STEP1 : For $n=1$ $(*)$ is true, since

$$1=1^2.$$

STEP2 : Suppose $(*)$ is true for some $n=k\geq 1$, that is

$$1+3+5+\ldots+(2k-1)=k^2.$$

STEP3 : Prove that $(*)$ is true for $n=k+1$:

$$1+3+5+\ldots+(2k-1)+(2k+1)\stackrel{?}{=}(k+1)^2.$$

We have :

$$1+3+5+\ldots+(2k-1)+(2k+1)$$

$$\stackrel{\text{ST.2}}{=} k^2+(2k+1)$$

$$=(k+1)^2. \blacksquare$$

Theorem : **Prove that**

$$n! \leq n^n \qquad (*)$$

for any integer $n \geq 1$.

Proof :

STEP1 : For $n = 1$ $(*)$ **is true, since**

$$1! = 1^1.$$

STEP2 : Suppose $(*)$ **is true for some** $n = k \geq 1$, **that is :**

$$k! \leq k^k.$$

STEP3 : Prove that $(*)$ **is true for** $n = k + 1$

$$(k+1)! \overset{?}{\leq} (k+1)^{k+1}.$$

We have :

$$\begin{aligned}(k+1)! = k! \cdot (k+1) &\overset{\text{ST.2}}{\leq} k^k \cdot (k+1) \\ &< (k+1)^k \cdot (k+1) \\ &= (k+1)^{k+1}. \blacksquare\end{aligned}$$

Theorem : Prove that

$$3^{2n} - 1 \text{ div. by } 8 \qquad (*)$$

for any integer $n \geq 0$.

Proof :

STEP1 : For $n = 0$ $(*)$ is true, since

$$3^0 - 1 \text{ div. by } 8.$$

STEP2 : Suppose $(*)$ is true for some $n = k \geq 0$, that is

$$3^{2k} - 1 \text{ div. by } 8.$$

STEP3 : Prove that $(*)$ is true for $n = k + 1$:

$$3^{2(k+1)} - 1 \overset{?}{\text{div. by}} 8.$$

We have :

$$3^{2(k+1)} - 1 = 3^{2k} \cdot 9 - 1 = \underbrace{3^{2k} \cdot 8}_{\text{div. by } 8} + \underbrace{3^{2k} - 1}_{\substack{\text{St. } 2 \\ \text{div. by } 8}}.$$

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