# M343K - Introduction To Algebraic Structures - Spring 2003

## Final Exam, May 10, 2003

**I. (20 points)** Prove that there are infinitely many prime numbers. **Show all steps and provide the necessary explanations.**

**II. (20 points)** Prove *algebraically* that if $p$ is a prime and $a \in \mathbb{Z}$, then $a^p \equiv a \mod p$. **Provide as many details as possible.**

**III. (20 points)**
Use mathematical induction to prove that:

(a) $\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 1 - \frac{1}{n}$, where $n \geq 2$;

(b) $2^n < n!$, where $n \geq 4$.

**IV. (20 points)**

(a) Find (120, 144) using the Euclidean Algorithm.

(b) Prove that $\sqrt[7]{3}$ is irrational.

**V. (20 points)** Let $\alpha = (1234)(56)$, $\beta = (123)(456)$. Find:

(a) $\alpha\beta$;

(b) $\beta\alpha$;

(c) $\beta^{-1}$;

(d) order of $H = \langle\alpha\rangle$;

(e) order of $H = \langle\beta\rangle$.

**Explain you work and justify answers everywhere.**

**VI. (20 points)**

1. Use the Euclidean algorithm to find the gcd of

$$x^7 - 1 \quad \text{and} \quad x^3 - 1$$

in $\mathbb{Z}$. **Show all steps and explain you work.**

2. Use the Euclidean algorithm to find the gcd of

$$x^3 + x - 2 \quad \text{and} \quad x^2 - 1$$

in $\mathbb{Z}_3$. **Show all steps and explain you work.**

**VII. (20 points)**

Show that there exist two nonisomorphic finite groups of the same order (**give an example and prove that they are in fact nonisomorphic**).

**VIII. (20 points)** Give examples of:
(a) Nonabelian finite group;
(b) Nonabelian infinite group;
(c) Finite cyclic group;
(d) Finite cyclic group with at least 100 different generators;
(e) Infinite cyclic group;
(f) Finite group with at least 100 different subgroups.
**Provide all the necessary explanations and justify answers.**

**IX. (20 points)** Give examples of:

(a) Commutative infinite ring;

(b) Commutative finite ring;

(c) Ring which is not a domain;

(d) Domain which is not a field.

**Provide all the necessary explanations and justify answers.**

**X. (20 points)** Let $R$ be a commutative ring. Show that
(a) $0a = 0$ for any $a \in R$;
(b) $(-1)(-a) = a$ for any $a \in R$;
(c) $(-1)a = -a$ for any $a \in R$.
**Show all steps and provide the necessary explanations.**