DEFINITION: If $c_0, c_1, c_2, \ldots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k = c_0+c_1(x-x_0)+c_2(x-x_0)^2+\ldots$$

is called a **power series in $\boldsymbol{x-x_0}$.**

EXAMPLE:

1. $$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots$$

2. $$\sum_{k=0}^{\infty} (-1)^k \frac{(x+2)^k}{k!} = 1-(x+2)+\frac{(x+2)^2}{2!} - \frac{(x+2)^3}{3!} + \ldots$$

THEOREM: For a power series $\sum c_k(x - x_0)^k$, exactly one of the following is true:

(a) The series converges only for $x = x_0$.

(b) The series converges absolutely (and hence converges) for all real values of x.

(c) The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R, x_0 + R)$, and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

EXAMPLE: Find the interval of convergence of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$.

SOLUTION: By the ratio test for absolute convergence we have

$$\begin{aligned}
\rho = \lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| &= \lim_{k\to\infty} \left|\frac{(x-5)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-5)^k}\right| \\
&= \lim_{k\to\infty} \left[|x-5| \left(\frac{k}{k+1}\right)^2\right] \\
&= |x-5| \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 \\
&= |x-5|
\end{aligned}$$

Thus, the series converges absolutely if $|x-5| < 1$, or $-1 < x-5 < 1$, or $4 < x < 6$. The series diverges if $x < 4$ or $x > 6$.

Now we examine the endpoints. If $x = 6$, then

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(6-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which converges by the p-series test. Similarly, if $x = 4$, then

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(4-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which converges by the alternating series test. So, the interval of convergence is $[4, 6]$ and the radius of convergence is $R = 1$.